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A

TREATISE

ON

ISOPERIMETRICAL PROBLEMS,

AND THE

CALCULUS OF VARIATIONS.

BODLEIANA BIBLIOTHECA

BY ROBERT WOODHOUSE, A.M. F.R.S.

FELLOW OF CAIUS COLLEGE, CAMBRIDGE.

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1810.

BIBLIOTHECA·BODLEIANA·

PREFACE.

There needs, perhaps, no other apology for the present Work, than the mere statement of the fact, that there is, on the same subject, no English, and only one Foreign Treatise *, of which the celebrated Euler is the Author.

The copies of this last, a work of uncommon merit, are in this country very rare. But, notwithstanding its various excellencies, a mere translation of it would not have rendered unnecessary the present treatise, since, independently of any other consideration, several of Euler's most important processes are, by more concise ones, now superseded.

But although there is only one separate Treatise, the subject itself has been discussed in several analytical works, English as well as

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* Methodus inveniendi Lineas curvas proprietate maximi minimive gaudentes, Lausannæ, 1744.

Foreign. Our countrymen are accustomed to assign, in their treatises on fluxions, a chapter to the maxima and minima of curves; and, the foreign mathematicians consider the same subject under the head of "*Calcul des Variations.*"

Maclaurin, Simpson, Emerson, especially the two latter, have not explained the principles of the subject with sufficient perspicuity and precision; and, in point of depth of research, there are various problems, to the solution of which, their formulæ are totally inadequate. Indeed, which is remarkable, there is, I believe, no English treatise that mentions Euler's formula of solution; which, for the simplicity of its construction, and facility of application, is exceeded by none in the whole compass of analytical science.

The foreign mathematicians, Euler, Le Seur, Bossut, and Lacroix, have furnished their treatises on the differential and integral calculi, with formulæ of solution adequate to any case that can be proposed. There is no deficiency on that head. But, these Authors too suddenly carry the reader into the middle of the business and immerse him in calculations; and, if they soon provide him with the instruments of solution, they instruct him neither in the object nor the principle of their construction.

This is not said in the spirit of an Author preparing, by the censure of preceding Works, a reception for his own. For the truth of the statement, I would appeal to the experience of all who have consulted the above cited Authors; besides, it is not difficult to assign the reason of the alledged defects.

When Lagrange in 1760, published his new method of solving problems of maxima and minima, he composed his memoir for mathematicians, familiar with its subject, and well versed in the researches of the Bernoullis and of Euler. Accordingly, he very briefly states the principles of his calculus, and enters into no explanation of the nature of the subject. His compendious method of computation, however, has been adopted; and subsequent Authors have composed their treatises very much on the plan of Lagrange's memoir, with some, but slight and imperfect, preliminary explanation. These Treatises, however, the student is expected to understand; that is, if the matter be fairly stated, he is expected to understand an intricate subject, with advantages much less than consummate mathematicians before him enjoyed; since, there is neither proper explanations presented to him, nor is he directed, by way of preparation, previously to consult the Works of Euler and the Bernoullis.

Such are, in my opinion, the defects of existing methods; still, however, I have not composed a treatise on the subject, by merely remedying them; that is, by inserting formulæ of sufficient extent, and by more fully explaining and illustrating their principles. But, on a novel plan, I have combined the historical progress with the scientific developement of the subject; and endeavoured to lay down and inculcate the principles of the Calculus, whilst I traced its gradual and successive improvements.

If this has been effected, which I think it has, in a compass not very wide of that which a strictly scientific treatise would have required, the only serious objection against the present plan is, in part, obviated. For, there is little doubt, the student's curiosity and attention will be more excited and sustained, when he finds history blended with science, and the demonstration of formulæ accompanied with the object and the causes of their invention, than by a mere analytical exposition of the principles of the subject, The plan, perhaps, would not suit any other department of science, so well as it does this; which is limited in its extent, and has had but few, although eminent cultivators.

Other advantages, besides that of an excited attention, may accrue to the student from the present plan. He will have an opportunity of observing how a calculus, from simple beginnings, by easy steps, and seemingly the slightest improvements, is advanced to perfection; his curiosity too, may be stimulated to an examination of the works of the contemporaries of Newton; works once read and celebrated: yet the writings of the Bernoullis are not antiquated from loss of beauty, nor deserve neglect, either for obscurity, or clumsiness of calculation, or shallowness of research. Their processes indeed are occasionally somewhat long, and want the trim form of modern solution. They are not, however, therefore the less adapted to the student, who is solicitous for just and full views of science, rather than for neat novelties and mere store of results. Indeed, the Authors who write near the beginnings of science, are, in general the most instructive: they take the reader more along with them, shew him the real difficulties, and, which is a main point, teach him the subject, the way by which they themselves learned it.

In a former Work *, I adopted the foreign notation, and the present occasion furnishes some proof of the propriety of that adoption. In the Calculus of Variations, it is necessary to have symbols denoting operations, similar to those that take place in the Differential Calculus: now, *d* being the symbol for the latter, δ is a most convenient one for the former: analogous to δ there is no symbol in the English system of notation. If then I had used the fluxionary notation with points or dots, I must have invented symbols corresponding to δ and the characters formed by means of it. But, the invention of merely new symbols is in itself an evil. M. Lagrange indeed, whose power over symbols is so unbounded that the possession of it seems to have made him capricious, has treated the subject of Variations without the foreign notation; this he rejects altogether; and, which is strange, has employed the English notation, but not adopted its signification. Thus, with him, $\dot{x}$ is not the fluxion, but the variation of x: the fluxions or differentials of quantities are not expressed by him, but solely the fluxionary or differential coefficients; thus, if u be a function of x, u' $\left(=\frac{du}{dx} \text{ or } = \frac{\dot{u}}{\dot{x}}\right)$ is the differential coefficient. What advantages are to arise from these alterations it is not easy to perceive: yet they ought to be great, to balance the plain and palpable evils of a confusion in the signification of symbols, and of the invention of a system of notation to represent what already was represented with sufficient precision. No authority can even sanction so capricious an innovation.

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* Principles of Analytical Calculation, 1803.

There is another point towards which I am not unwilling to draw the attention of the reader; and that is, the method of demonstration by geometrical figures. In the first solution of Isoperimetrical problems, the Bernoullis use diagrams and their properties. Euler, in his early essays, does the same; then, as he improves the calculus he gets rid of constructions. In his Treatise*, he introduces geometrical figures, but almost entirely, for the purpose of illustration: and finally, in the tenth volume of the *Novi Comm. Petrop.* as Lagrange had done in the *Miscellanea Taurinensea*, he expounds the calculus, in its most refined state, entirely without the aid of diagrams and their properties. A similar history will belong to every other method of calculation, that has been advanced to any degree of perfection.

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* Methodus inveniendi, &c.

The plan of the Work has been slightly described; and indeed it scarcely requires any explanation. On that, however, I chiefly rely, as the means of rendering easy a subject, which is acknowledged to be one of the most difficult. But, although I am not aware of having omitted any thing that is requisite to the full explanation of the subject, yet I cannot flatter myself that it will be thoroughly understood from this Work alone. For, in general it may be laid down as true, that no doctrine, of novelty and intricacy, can be completely taught by a single Treatise. It seems to be indispensably necessary for the student, that the subject should be put under several points of view: that if not apprehended under one, it may be under another. For this reason, though not wanting an Author's partiality for his own performance, I recommend the perusal of those works, to which frequent reference is made in the following pages.

I return my sincere thanks to the Syndics of the University Press, who, very liberally, have defrayed two-thirds of the expense of the present publication.

Caius College,
Nov. 17, 1810.

Table *of the* Foreign, *and the corresponding* English *Notation.*

F.	dx	d^2x	d^3x	d^nx	dx^2	dx^3
E.	$\dot{x}$	$\ddot{x}$	$\dddot{x}$	$\overset{\mathrm{n}}{\dot{x}}$	$\dot{x}^2$	$\dot{x}^3$
F.	$\frac{du}{dx}$	$\frac{d^2u}{dx^2}$	$\frac{d^3u}{dx^3}$	$\frac{d^2u}{dx.dy}$	$\frac{d^3u}{dx^2.dy}$	$d^3\left(\frac{dy}{dz}\right)$
E.	$\frac{\dot{u}}{\dot{x}}$	$\frac{\ddot{u}}{\dot{x}^2}$	$\frac{\ddot{u}}{\dot{x}^3}$	$\frac{\ddot{u}}{\dot{x}\dot{y}}$	$\frac{\dddot{u}}{\dot{x}^2\dot{y}}$	$\left(\frac{\dot{y}}{\dot{z}}\right)^{\therefore}$

List of Foreign Authors that treat of the Subject of the present Work.

Bernoulli, James and John. — Works.

Euler.
- Comm. Acad. Petrop. tom. VI. VIII. 1733, 1736.
- Methodus Inveniendi Lineas curvas proprietate maximi minimive gaudentes, *Lausanne. Geneva*, 1744.
- Novi Comm. Petrop. tom. X. 1766.
- Calcul Integral, 1768.

Lagrange.
- Miscellanea Taurinensia, tom. II. IV.
- Théorie des Fonctions Analytiques, 1797.
- Leçons sur le Calcul des Fonctions, 1806.

Borda, et **Fontaine**.—Acad. des Sciences, *Paris*, 1767.
Le Seur et **Jacquier**.—Elemens de Calcul Integral, 1768.
Sauri.—Cours de Mathematiques, tom. V. 1774.
Legendre.—Acad. des Sciences, *Paris*, 1786.
Lacroix.—Calc. Diff. et Integral, 1797.
Bossut.—Calc. Diff. et Integral, 1798.

ON

ISOPERIMETRICAL PROBLEMS,

AND THE

CALCULUS *of* VARIATIONS.

CHAP. I.

The Problem of the Curve of Quickest Descent proposed.—James Bernoulli's Solution of it.—Principle of that Solution.

THE ordinary questions of maxima and minima were amongst the first that engaged the attention of mathematicians at the time of the invention of the Differential or Fluxionary Calculus. This invention took place about the year 1684, three years before the memorable æra of the publication of the Principia. But, the principles of the differential calculus, were not, like those of physical astronomy, given to the world, at once, and as it were, on a sudden, in a formal treatise. They were communicated gradually and by piecemeal; in letters between men of science, in small essays and tractates, in solutions of particular problems published chiefly in a work intitled the *Acta Eruditorum* edited at Leipsick *; which, at that time, was the common vehicle of communication between men of science and literature. There is scarcely a formula or process, which we now find compactly stated in our modern treatises, that was not in those Acts the subject of some correspondence and discussion. The methods of drawing tangents, radii of curvature, of determining the points of inflexion, the maxima and minima of quantities, &c. were treated as subjects of great importance by Leibnitz, and by the two Bernoullis.

The first problem relative to a species of maxima and minima distinct from the ordinary, was proposed by Newton in the Principia: it was, that of the solid of least resistance. But, the subject and doctrine became not matter of discussion and controversy, till John Bernoulli, Professor of mathematics at Groningen, required of mathematicians the determination of the curve of quickest descent. This he did in the *Leipsick Acts* for June, 1696 †, p. 269, under the following form:

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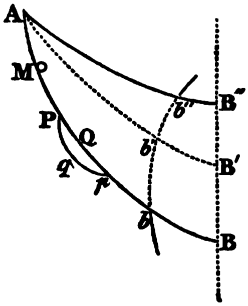
* Problems proposed and solved, became an amazing means of promoting the methods of calculation, or the calculus, to which they belonged. Of this John Bernoulli was aware, when, in his famous Programma relative to the line of quickest descent, he says, "Cum compertum habeamus, vix quicquam esse quod magis excitet generosa ingenia, ad moliendum quod conducit augendis scientiis, quam difficilium pariter, et utilium quæstionum propositionem." *Opera*, tom. I. p. 166.

† Opera, tom. I. p. 155.

"PROBLEMA NOVUM,

ad cujus Solutionem Mathematici invitantur.

"Datis in plano verticali duobus punctis *A* et *B*, "assignare mobili *M* viam *AMB*, per quam gravitate



"sua descendens, et moveri incipiens a puncto *A*, brevissimo tempore perveniat ad alterum punctum *B*."

Six months, the time allotted for its solution, being elapsed, and no solution appearing, John Bernoulli at the request of Leibnitz (who intimated that he had solved the problem) prorogued the term, and again anounced the problem in a *programma** edited at Groningen in January 1697. In the May following, his brother, James Bernoulli, professor of mathematics at Basle, published its solution in the *Acta Erud. Lips.* p. 211†; and after the following manner:

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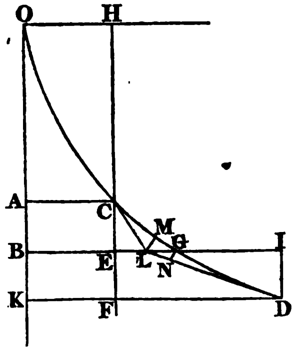
* Opera, tom. I. p. 166.

† See also his Works, tom. II. p. 768.

Let OGD be the curve; conceive a portion of it CGD, to be divided into two parts CG, GD; and take another



element to the curve CLD, divided also into two parts CL, LD, and indefinitely near to CGD: then since by hypothesis, the time through $CG+GD$, is to be a minimum, and since quantities at or near their state of minimum may be considered constant (for their increments or decrements are very small), we have

$$t.CG+t.GD=t.CL+t.LD$$

[$t.CG$, abridgedly representing the time through CG.]

and $\therefore$ $t.CG-t.CL=t.LD-t.GD$

again $CE:CG::t.CE:t.CG$

[CG considered as an inclined plane.]

and $CE:CL::t.CE:t.CL$

consequently, $CE:CG-CL\ [MG]::t.CE:t.CG-t.CL$

but $MG:LG::EG:CG$

[by similar Δs LMG, GCE.]

$\therefore CE:LG::EG\times(t.CE):CG\times(t.CG-t.CL)$

similarly $EF:LG::GI\times(t.EF):GD\times(t.LD-t.GD.)$

Hence, equating the two values of LG, we have

$$EG \times t.CE \times EF \times GD = GI \times t.EF \times CE \times CG$$

$$\text{or } EG \times \frac{CG}{\sqrt{HC}} \times EF \times GD = GI \times \frac{GD}{\sqrt{HE}} \times CE \times CG$$

[substituting for $t.CE$, $t.EF$.]

$$\text{or } \frac{EG}{\sqrt{HC}} : \frac{GI}{\sqrt{HE}} :: CE : EF :: CG : GD \quad [1.]$$

which is a property of the cycloid*: hence the curve of quickest descent, or the *brachystochrone*, is a cycloid.

The proportion [1] may be thus expressed:

$$\frac{EG}{GC} : \frac{GI}{GD} :: \sqrt{HC} : \sqrt{HE}, \qquad \text{or}$$

sin. $\angle ECG$: sin. $\angle GDI$ :: $\sqrt{HC}$: $\sqrt{HE}$ :: vel^y. at C : vel^y. at G, or the sine of the angle made by a vertical and an element of the curve is proportional to the velocity.†

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* This will easily appear by constructing the figure with its generating circle; or, the known equation of the cycloid may be thus deduced: since $\frac{EG}{\sqrt{HC.CG}} = \frac{GI}{\sqrt{HE.GD}}$ we have $\frac{dx}{\sqrt{y}.dz} = \frac{dx'}{\sqrt{y'}.dz'}$ and since dx', y', dz', are the consecutive values of dx, y, dz, it follows from the above equation, that $\frac{dx}{\sqrt{y}.dz}$ is every where the same, or is constant, and therefore may be put $= \frac{1}{\sqrt{a}}$. Hence, $a\,dx^2 = y\,.\,dx^2 + y\,.\,dy^2$ and $dx = \frac{\sqrt{y}}{\sqrt{(a-y)}}dy$, the equation to a cycloid.

† Lagrange, *Cal. des Fonct.* 8vo. p. 425, says that this was the form of the result obtained by Bernoulli. It is not however the form given in the first solution that occurs in Bernoulli's Works.

In the preceding demonstration two principles are involved: one* borrowed from the doctrine of the ordinary maxima and minima of quantities: the other, at the time of the first discussion of these questions entirely new, and in fact assuming; That the property appertaining to the whole curve, belongs also to any element of the curve: for of the preceding problem, the condition was, that the time down the whole curve *OCD* should be a minimum, that is, less than down any other curve, passing through the points *O*, *D*: but in the demonstration, an element *CGD* is taken, and it is assumed that the time down this element is also a minimum: so that, the whole curve is determined, by determining a portion or element of it. The principle undoubtedly is true, both in the preceding and in other problems, but it is not universally true. Of the exceptions, and of their reasons, we shall hereafter speak, and here only briefly notice, that the Bernoullis and Brook Taylor, preface their small tracts on this subject, with this principle, establishing the same nearly after the following manner:

If the curve *APB* (see fig. p. 3.) has a certain property of minimum, a portion of it *PQp*, has the same property: for suppose *AP*, *Bp*, to be determined, then if *PQp* possesses not the property of maximum or minimum, suppose another small portion *Pqp* to possess it; therefore *Pqp* added to *AP*, and *Bp*, will form a curve, that in the case of maximum will contain more, or of minimum, less than $APQpB$; which is contrary to the hypothesis*.

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* See page 4. line 5, &c.

The first principle is, as it has been remarked, that which is employed in the ordinary questions of maxima and minima. If, for instance, we wish to determine when an ordinate y is a minimum, dy its differential is put equal to nothing. Hence, the contiguous or consecutive ordinate which generally is,

$$y \pm \frac{dy}{dx} . \Delta x + \frac{d^2y}{2.dx^2} (\Delta x)^2 \pm \text{\&c.}\dagger$$

is in this case reduced to

$$y + \frac{d^2y}{2.dx^2} . (\Delta x)^2 \pm \text{\&c.}$$

which, if we make Δx infinitely small, will differ from the ordinate y only by infinitely small quantities of the second order; and if we neglect them, may be said to be equal to the ordinate y. Of this nature, and requiring this explanation, is the equality in James Bernoulli's demonstration, when the time through CG + the time through GD = the time through CL + the time through LD.

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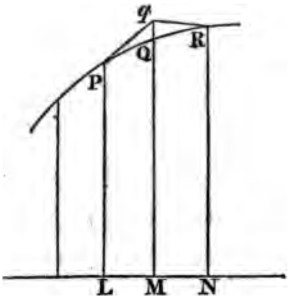
* P. 226. tom. II. John Bern. *Leip. Act.* 1701, p. 213. James Bernoulli, the author of the essay, as if suspicious of the truth of his principle, says, "Sensus theorematis vel demonstrationis ejus videtur paulo obscurior, nec satis determinatus: sed planior et infra ex applicatione, &c." John Bernoulli also, *Acad. des Sciences*, 1706, p. 236. lays down the same principle: "Parceque toute courbe qui doit donner un maximum, conserve aussi dans toutes ses parties les loix de ce même maximum, &c."

† Woodhouse, *Prin. Analy. Calc.* pp. 44, 163.

The use of the principle, however, in Bernoulli's problem, and its illustration, will enable us to point out the real and essential distinction between the ordinary calculus of maxima and minima, and that which is the subject of the present treatise. In the former the relation of y to x is supposed to be given, or, in other words, the function f in the equat^n. $y = fx$ is supposed to be known; and the process of equating y and $y \pm \frac{dy}{dx} . \Delta x + \&c.$, or of making $dy = 0$, enables us to find a particular value of x which substituted in $y = fx$ gives the greatest or least value of y. Thus in the equation $y = \sqrt{(2ax - x^2 + b)}$, $fx = \sqrt{(2ax - x^2 + b)}$ and if the greatest ordinate



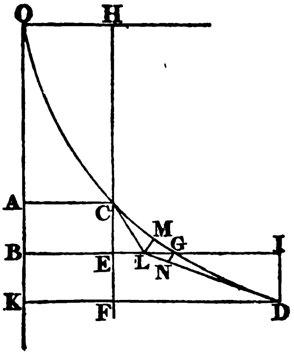
be required, (putting $MQ = y$, and $MN = \Delta x$), $NR = y + \frac{dy}{dx} . \Delta x . + \frac{d^2 y}{2 . dx^2}(\Delta x^2) + \&c.$ and $\frac{dy}{dx} = \frac{a - x}{y} = 0$, consequently $x = a$, and the maximum value of $y = \sqrt{(a^2 + b)}$. In the latter calculus, on the contrary, the relation of y to x is not given but sought, or in other words the form of the function fx [that is, whether it is $\sqrt{(2ax - x^2)}$ or e^{ax} or $(ax^n - bx^m)^{\frac{1}{r}}$, &c.] is the object of investigation.

For this reason, the augmented value of $y = MQ$ is not, as before, its consecutive value $NR = y + \Delta y$, but $Mq = y + Qq = y + v$, which must be substituted for y in expanding any function of y. Thus, if the equation between y and x, or in other words, if the function fx is required, such that $\int (ax - y^2)\, y . dx$ shall be a maximum, we have, changing y into $y + v$, the consecutive value of the expression for the maximum, $(axy - y^3) + (ax - 3y^2)v +$ &c. and since this expression must be a maximum, the second term, analogous to $\frac{dy}{dx}\Delta x$ in the former instance, must $= 0$, or $ax - 3\,y^2 = 0$, and $y = \sqrt{\frac{ax}{3}}$.

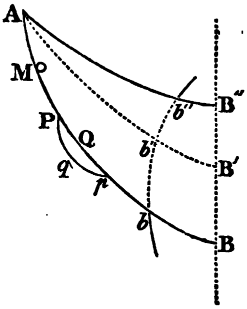
Here it is plain, the result gives the *form* of the equation, that must subsist between y and x, so that generalizing, it may be said, that it is the form of the function fx in $y = fx$, which is the object of search. The two preceding instances plainly shew the points of agreement and disagreement of the two calculi.

Before we quit this part of our subject we will shew how Bernoulli's method of solution, without any alteration of principle, may be abridged. Since the time varies as the space divided by the velocity, and the velocity as the square root of the height, we have,

$\frac{CL}{\sqrt{HC}} + \frac{LD}{\sqrt{HE}} = \frac{CG}{\sqrt{HC}} + \frac{GD}{\sqrt{HE}}$, from the principle of minimum, $\therefore \frac{LC - (CL + MG)}{\sqrt{HC}} = \frac{(LD - LN) - LD}{\sqrt{HE}}$, or $\frac{MG}{\sqrt{HC}} = \frac{LN}{\sqrt{HE}}$, or from similar triangles, $\frac{EL}{CL\sqrt{HC}} . LG = \frac{LI}{LD\sqrt{HE}} . LG$, or $\frac{EL}{CL\sqrt{HC}} = \frac{LI}{LD\sqrt{HE}}$ as before, p. 5.



In Bernoulli's solution the curve between *A* and *B* is shewn to be a cycloid; but nothing farther is determined; the result is independent of the relative position



of the points *A* and *B*. Hence, the same result, as far as it regards the nature of the curve, will obtain for *A* and *B′*, *A* and *B″* &c. that is, the curves *AB*, *AB′*, *AB″*, &c. are all cycloids. A question then naturally arose out of the original one; Which is the cycloid, down whose length a body will descend in the least time? Or, in other words, required the curve, a cycloid it must be, such that a body shall descend from *A* to the vertical line *BB′B″* in the least time. This, after the solution of the first problem by James Bernoulli, was proposed by him to his brother, and solved by the latter, not only for the particular case in which *BB′B″* is vertical, but for any inclination of that line, and for any form of that line; that is, John Bernoulli determined the condition to which the cycloid must be subject, when the time down it from *A* to a curve *b b′ b″* shall be a minimum. The condition, which hereafter will be deduced, is, that the cycloid shall cut the curve at right angles *.

We may now, without either haste or abruptness, proceed in the historical and scientific development of the new calculus. And, the next step will be to the famous Programma of James Bernoulli, which contained the problem, whence the title of Isoperimetrical, since applied to all problems of a like kind, is derived.

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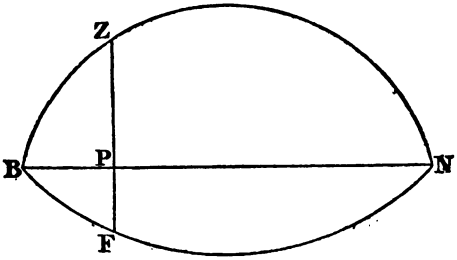
* This Bernoulli determined by an ingenious, but peculiar method. Indeed, there appeared no general method of solving isoperimetrical problems, with all their circumstances, till Lagrange in 1760 gave his formula of solution, consisting of two parts; the one containing integrals, the other a definite expression. See the subsequent parts of this Work.

CHAP. II.

erimetrical Problems proposed by James Bernoulli — Solved by ohn Bernoulli—His fundamental and specific Equations — Application of them to the Curve of Quickest Descent, and of a given Length—Solution of Isoperimetrical Problems, by Brook Taylor— Imperfections of his and the Bernoullis's Methods.

In 1697, James Bernoulli edited a Programma, which proposed this problem *:

Of all Isoperimetrical curves described on the common base *BN*, to find *BFN* such, that another curve *BZN*



shall contain the greatest space, the ordinate of which *PZ*, is in any multiplicate or submultiplicate ratio of the ordinate *PF*, or of the arc *BF*.

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* "Quæritur ex omnibus isoperimetris, super communi basi *BN* constitutis, illa *BFN*, quæ non ipsa quidem maximum comprehendat spatium, sed faciat, ut aliud curva *BZN* comprehensum sit maximum, cujus applicata *PZ* ponitur esse in ratione quavis multiplicata, vel submultiplicata, rectæ *PF*, vel arcus *BF*, hoc est, quæ sit quotacunque proportionalis ad datam *Ag* rectam *PF*, curvamve *BF*." *Acta Erud.* 1697. Mai. p. 214.

In the terms of modern analysis, the ordinate *PZ* is said to be a function of *PF*, or a function of the arc *BF*.

It is the last case of the problem, when *PZ* is a function of the arc *BF*, that baffled the sagacity and skill of John Bernoulli, and caused between him and his brother, a long and acrimonious controversy. All the solutions of the former were erroneous[*], till the publication of his brother's. And, even when John Bernoulli had amended his solution, he would not frankly and plainly acknowledge his error, but slurred it over, with a faint and half-confession of having previously been guilty of some slight inadvertencies. The last solution, however, of John Bernoulli, published in the Academy of Sciences for the year 1718, is, considering what was then the state of analytical science, very admirable, and merits the eulogium which he himself has conferred on it, that of being equally exempt from the tediousness of his brother's, and the obscurity of Taylor's calculations[†]. It is, however, obviously borrowed from his brother's, which was entitled "Analysis magni Problematis Isoperimetrici," and published in the *Acta Eruditorum Lips.* for May, 1701, p. 213. Previously to the *Analysis*, a solution, as it was called, but in which results alone were stated, had been published in the above mentioned Acts, for June, 1700, p. 261.

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[*] *Acad. des Sciences*, 1706, p. 235. or *Opera*, tom. I. p. 424. In this memoir he solved the first case of his brother's problem, making two elements only of the curve to vary, but making besides the elements of the abscissa to vary. On these grounds the solution of the first case is right, but that of the second erroneous.

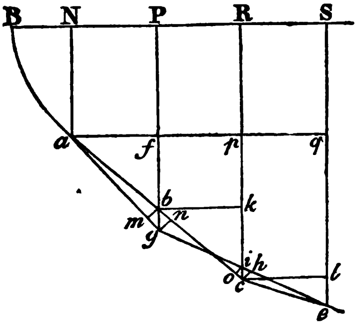
[†] "Une voye courte, claire et facile, suivant laquelle un geometre d'habiletè et d'esprit mediocres puisse arriver jusqu' avoir, non sur la foi d'autrui, mais de ses propres yeux, ces veritez abstruses, sans s'engager dans la longueur du calcul de mon frère, ni dans l'obscurité de celui de M. Taylor." *Opera*, tom. II. p. 238, also *Mem. de l'Acad. de Paris*, 1718. p. 100.

As this treatise is not intended to be strictly historical, the problem of the isoperimetricals will not be traced through all its variety of change and emendation. To history we shall adhere no farther, than is sufficient to preserve an unbroken series of methods gradually becoming more exact and extensive; the series beginning with the first rude, though perfectly just, method of James Bernoulli, and ending with Lagrange's exquisite and refined *Calculus of Variations*. For this reason we proceed to notice, and shall give the substance of the last solution of John Bernoulli, published in the Memoirs of the Academy of Sciences for 1718. The separation between that, and the first solution of the *brachystochrone*, which has been already given, being neither too wide, nor too abrupt, for our present purpose.

Bae is a curve, and an element of it $abce$ is composed of three portions ab, bc, ce; another element $aghe$, indefinitely near the former, is composed of three portions, ag, gi, ie, and since the solution is to extend to *isoperimetrical* curves, we have,

$$ab + bc + ce = ag + gi + ie,$$

and consequently, $(ag - ab) + (gi - bc) + (ie - ce) = 0$



$$\text{or, } gm - bn - oc + ih = 0 \qquad [1]$$

$$\text{By similar triangles, } \begin{cases} bmg,\ abf, & gm = \dfrac{fb}{ab} \cdot bg \\ bng,\ bck, & bn = \dfrac{kc}{bc} \cdot bg \\ ioc,\ bck, & co = \dfrac{kc}{bc} \cdot ci. \\ ich,\ cle, & ih = \dfrac{le}{ce} \cdot ci. \end{cases}$$

Hence, substituting in the equation [1]

$$\left(\frac{fb}{ab} - \frac{kc}{bc}\right) bg - \left(\frac{kc}{bc} - \frac{le}{ce}\right) ci = 0,$$

and this Bernoulli calls his fundamental equation, on account of the uniformity * subsisting in the coefficients

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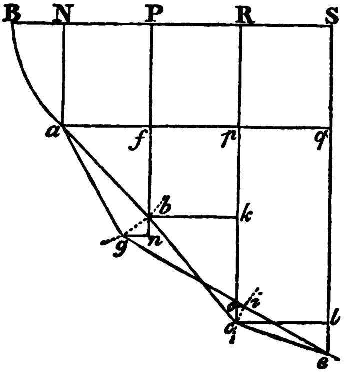
* P. 104. *Mem. Acad. Paris,* 1718. Bernoulli, after noting this uniformity and shewing what it consists in, adds, "On verra dans la suite que cette uniformité contribue merveilleusement a reconnoitre comme d'un seul coup d'œil et non en consequence d'aucune analyse, les equations qui conviennent a chacun des problêmes que nous allons resoudre."

of bg and ci; which uniformity, if we run a little before our matter, and borrow the symbols of the differential calculus, may be easily made manifest. Thus, fb, kc, le, being the differentials of the ordinates Na, Pb, Rc [y, y', y''] are to be noted, dy, dy', dy''; and if dz, dz', dz'' denote the differentials of the three arcs Ba, Bb, Bc, we have,

$$\left(\frac{dy}{dz}-\frac{dy'}{dz'}\right) bg = \left(\frac{dy'}{dz'}-\frac{dy''}{dz''}\right) ci.$$

In the foregoing demonstration, the elements of the abscissa are supposed to be constant, or the points b, c, by whose motion the curve is supposed to be changed, move along the lines fg, pc; but, instead of the elements of the abscissa, those of the curve may be supposed invariable, and this case Bernoulli includes in a second fundamental equation, like the former, and demonstrated by the subjoined process.

Since the elements ab, bc, ec, are supposed to be



constant, from a, e, as centers describe circular arcs

bg, ci, in which, the points b, c, when we pass from curve to curve, are supposed to move: now,

$$bc^2 = gi^2$$

$$\therefore\ bk^2 + kc^2 = (bk+gn+oi)^2+(kc-co-bn)^2$$

or, $(bk+gn+oi)^2-bk^2=kc^2-(kc-co-bn)^2$

or, by virtue of the formula,

$$x^2-y^2=(x+y)\ (x-y)$$

$$(2bk+gn+oi)\ (gn+oi)=[2kc-(co+bn)]\ (co+bn)$$

or, $2bk\ (gn+oi)=2kc\ (co+bn)$ neglecting $(gn+oi)^2$, $(co+bn)^2$; hence, $gn+oi=(co+bn)\ \frac{kc}{bk}\ldots[2]$

now, by similar triangles, $\begin{cases} afb,\ bng, & gn=\frac{fb}{af}.bn \\ cle,\ coi, & oi=\frac{le}{cl}.co. \end{cases}$

Hence, substituting in equation [2], and transposing

$$\left(\frac{fb}{af}-\frac{kc}{bk}\right)bn-\left(\frac{kc}{bk}-\frac{le}{cl}\right).co=0,$$

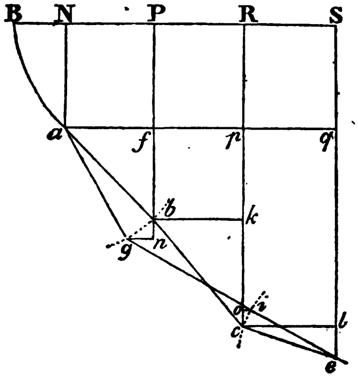
which Bernoulli calls also a fundamental equation, and in which is to be noted an uniformity similar to that pointed out in the former fundamental equation.

If we express the equation symbolically, we shall have,

$$\left(\frac{dy}{dx}-\frac{dy'}{dx'}\right)bn-\left(\frac{dy'}{dx'}-\frac{dy''}{dx''}\right).co=0.$$

These equations appeared in Bernoulli's last solution of the Isoperimetrical Problems. And to be satisfied of the skill and previous labour expended in their construction, we need only remark, that they have no dependance on any property of maximum, or minimum, and seem therefore to have no relation to the problems, for which they were specially designed. The relation, however, which they are made to have, cannot be better understood than from the following instances. The first of which, is the ***Brachystochrone,*** not the common case*, but that, in which, to the minimum property, the Isoperimetrical is added. In other words, when, out of all curves of a given length, the curve of quickest descent is required.

In the subjoined figure let $Babce$ be the curve of



quickest descent, of which a portion $abce$ is formed of three elements, ab, bc, ce; take a portion of another curve formed of ag, gi, ie, indefinitely near the former, then, by the principle of the maxima and minima of quantities, the time down ab, bc, ce, = time down ag, gi, ie. But the time varies as the space divided by the velocity varies; therefore as the element of the curve divided by the square root of the vertical height from which the body has fallen: hence,

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* See pages 4, and 9.

$$\frac{ab}{\sqrt{Na}}+\frac{bc}{\sqrt{Pb}}+\frac{ce}{\sqrt{Rc}}=\frac{ag}{\sqrt{Na}}+\frac{gi}{\sqrt{Pn}}+\frac{ie}{\sqrt{Ro}}$$

but ab, ag, bc, &c. are all equal;

$$\therefore \frac{1}{\sqrt{Pb}}+\frac{1}{\sqrt{Rc}}=\frac{1}{\sqrt{Pn}}+\frac{1}{\sqrt{Ro}},$$

$$\text{or, } \frac{1}{\sqrt{Pb}}-\frac{1}{\sqrt{Pn}}=\frac{1}{\sqrt{Ro}}-\frac{1}{\sqrt{Rc}},$$

$$\text{or, } \frac{1}{\sqrt{Pb}}-\frac{1}{\sqrt{(Pb+bn)}}=\frac{1}{\sqrt{(Rc-co)}}-\frac{1}{\sqrt{Rc}},$$

$$\text{or, } \frac{1}{\sqrt{Pb}}-\left(\frac{1}{\sqrt{Pb}}-\frac{bn}{2Pb^{\frac{3}{2}}}\right)=\frac{1}{\sqrt{Rc}}+\frac{co}{2Rc^{\frac{3}{2}}}-\frac{1}{\sqrt{Rc}}$$

[expanding and neglecting the terms involving bn^2, co^2, &c.]

Hence, $\frac{bn}{co}=\frac{Pb^{\frac{3}{2}}}{Rc^{\frac{3}{2}}}$; and this is an equation expressing the proportion of bn to co, obtained from the property of the proposed minimum; but the second fundamental equation, involves bn, co, the proportion of bn to co, there arising from the *Isoperimetrical* condition. Hence, eliminating bn, co, there results the specific equation*

$$\left(\frac{fb}{af}-\frac{kc}{bk}\right)Pb^{\frac{3}{2}}-\left(\frac{kc}{bk}-\frac{le}{lc}\right)Rc^{\frac{3}{2}}=0,$$

$$\text{or, } \left(\frac{fb}{af}-\frac{kc}{bk}\right)Pb^{\frac{3}{2}}=\left(\frac{kc}{bk}-\frac{le}{lc}\right)Rc^{\frac{3}{2}}$$

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* "Specifique; parceque d'elle resulte l'equation differentielle, qui determine finalement l'espece de la courbe cherchée." *Acad. des Sciences*, 1718, p. 108.

Now if we observe this equation, it will appear perfectly uniform in its two members, since the right hand member differs from the left, only inasmuch as the points *e*, *c*, are used instead of the *c*, *b*. Hence, either member is always the same, throughout the curve required: hence, symbolically,

$$\left(\frac{dy}{dx}-\frac{dy'}{dx'}\right) y^{\frac{3}{2}} = \text{a constant quantity;}$$

or, $-d\left(\frac{dy}{dx}\right) y^{\frac{3}{2}} = \surd a \,.\, dz$

[dz is constant and multiplied into $\surd a$, in order to make the equation uniform.]

that is, $\frac{d^2x \,.\, dy - d^2y \,.\, dx}{dx^2} = \surd a \,.\, \frac{dz}{y^{\frac{3}{2}}} \; . \; . \; . \; [a]$

but $dz^2 = dx^2 + dy^2$; and since dz is invariable, $o = dx \,.\, d^2x + dy\, d^2y$; hence, substituting in [a]

$$\frac{d^2x\,(dx^2+dy^2)}{dx^2} = \surd a \,.\, \frac{dz \,.\, dy}{y^{\frac{3}{2}}}$$

or, $\frac{d^2x}{dx^2} \,.\, dz = \surd a\, \frac{dy}{y^{\frac{3}{2}}}$, or $d\left(\frac{1}{dx}\right) \,.\, dz = 2\surd a \,.\, d\left(\frac{1}{y^{\frac{1}{2}}}\right)$

and integrating, $\frac{dz}{dx} = 2\sqrt{\frac{a}{y}} \pm \frac{b}{c}$ [$\frac{b}{c}$ being the correc^{n}*.]

Hence, $\frac{dz^2}{dx^2}$, or $\frac{dx^2+dy^2}{dx^2} = 4\frac{a}{y} \pm 4\frac{b}{c}\sqrt{\frac{a}{y}} + \frac{b^2}{c^2}$

and, $dx = \frac{cy^{\frac{1}{2}} \,.\, dy}{\surd(4ac^2 + b^2y - c^2y \pm 4bc\surd ay)}$.

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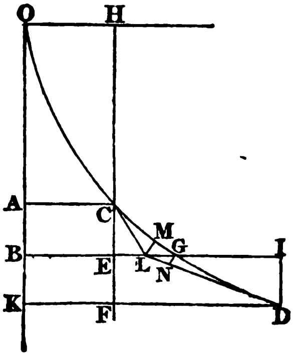
* See Emerson's Fluxions, p. 189. Third Edition.

If in the correction $\frac{b}{c}$, $b = 0$, $dz = 2dx\sqrt{\frac{a}{y}} = 2\sqrt{a}.\frac{dy}{\sqrt{(4a-y)}}$, and the curve is the common cycloid; which now is the curve of the quickest descent, not amongst all curves of the same length between *B* and *C*, but amongst all curves whatever that can be drawn between these two points. This is, however, only a particular case, for if *b* be real, the curve is not a cycloid, nor is it any known curve, but merely that which the differential equation determines it to be.

If we now examine the preceding solution, we shall perceive the office of the fundamental equations in solving problems of maxima and minima. Analytically speaking, they half resolve such problems; and furnish one equation involving two arbitrary quantities, [*bn*, *co*] whilst the maximum or minimum property furnishes another. That which results from elimination is the equation of solution.

We may perceive also, in the preceding solution, the necessity of making three elements of the curve to vary. Two elements are sufficient when merely the *brachystochrone* without any restriction is required. The minimum property furnishes one equation, containing an arbitrary quantity [*LG*], which is also the equation of solution. If a second property, isoperimetrical, or other, be introduced, it may be incompatible with the curve determined by the equation resulting from the minimum property.

For instance, *CGD* may be determined such, that it



is the curve of quickest descent, and then taking *CLD* indefinitely near, the time down *CG*+the time down *GD*=the time down *CL* + the time down *LD*; but *CG*+*GD* cannot equal *CL*+*LD*, for the former sum is less than that of any two lines, drawn from the extremities *C*, *D*, externally to *CG*, *GD*.

This does not shew, beyond controversy, the necessity of the variation of three elements. For, the points *L*, *G*, need not lie in the same line *ELG*, since the elements of the abscissa as well as those of the curve may be made to vary: in which case, two equations, derived from two properties may be obtained; and this method, which was used by John Bernoulli *, in certain cases, but not generally, leads to right results.

If problems involving merely one property, the maximum, require the variation of two, and those involving two properties, the variation of three elements; problems involving three properties, would, it should seem, require the variation of four elements: and this is the case. Each property furnishes an equation containing three arbitrary quantities (such as bn, co), and the solution depends on the equation resulting from their elimination. To such problems, however, the Bernoullis did not extend their researches.

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* The method is right in the first case of the problem proposed by James Bernoulli, but not, in the second. See p. 13.

We shall give a second instance of Bernoulli's method in solving the first Isoperimetrical problem that was proposed by his brother; and proceed as in the former case; that is, use two fundamental equations involving bg, ic, one derived from the Isoperimetrical condition, the other from the common property belonging to quantities, at their state of maximum. Now the problem requires the curve to be determined, such, that if y be the ordinate, and $Y = f(y)$*, $\int Y\,dx$ shall be a maximum.

Hence, since the elements of the abscissa are supposed constant, [see fig. p. 15.]

$$f(Pb) + f(Rc) = f(Pg) + f(Ri),$$

by the property of maximum,

$$\text{or, } f(Pb) - f(Pg) = f(Ri) - f(Rc)$$

$$\text{or, } f(Pb) - f(Pb + bg) = f(Rc - ci) - f(Rc)$$

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* $f(y)$ is the method of noting the function of a quantity y. The name and symbol were used by John Bernoulli in the Memoirs of the Academy for 1706, and 1718. See the latter Vol. p. 106.

from which, expanding $f(Pb + bg)$, $f(Rc - ci)$*, and neglecting the terms involving the squares, cubes, &c. of bg, ci, we have,

$$\mathrm{D}.f(Pb).bg = \mathrm{D}.fRc.ci,$$

[$\mathrm{D}.f(Pb)$ being the coefficient of the second term, in the expansion of $f'(Pb + bg)$†]

If we now substitute for bg, ci, in the fundamental equation [1] p. 15, there results,

$$\left(\frac{fb}{ab} - \frac{kc}{bc}\right) \frac{1}{\mathrm{D}.f'(Pb)} = \left(\frac{kc}{bc} - \frac{le}{ce}\right) \frac{1}{\mathrm{D}.f'.(Rc)}.$$

In which equation, (specific Bernoulli calls it) the law of uniformity being observable in the composition of its members, it follows, that,

$$\left(\frac{fb}{ab} - \frac{kc}{bc}\right) \frac{1}{\mathrm{D}.f'(Pb)},$$ is every where the same,

or, $$\left(\frac{dy}{dz} - \frac{dy'}{dz'}\right) \frac{1}{\mathrm{D}.Y}, \text{ or } -d\left(\frac{dy}{dz}\right) \frac{1}{\mathrm{D}.Y} = \frac{dx}{a},$$

$\frac{dx}{a}$ being from the invariability of the elements of the abscissa, constant;

$$\text{hence, } \frac{d^2z\, dy - d^2y\,.\,dz}{dz^2} = \frac{\mathrm{D}.Y.dx}{a},$$

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* $Pb = y$, and $bg = dy$, then, by Taylor's theorem, [See *Princ. Anal. Calc.* p. 44.]

$$f(y + dy) = fy + \frac{d.fy}{dy}.dy + \frac{d^2.fy}{2.dy^2}.dy^2 + \&c.$$

or, if we make $Y = fy$,

$$f(y + dy) = Y + \frac{dY}{dy}.dy + \frac{d^2Y}{2.dy}.dy^2 + \&c.$$

† *Princ. Anal. Calc.* Pref. p. xxix; and p. 43.

but $dz^2 = dx^2 + dy^2$; $\therefore dz \,.\, d^2z = dy \,.\, d^2y$; since dx is constant, consequently, by substitution,

$$\frac{d^2z\,(dy^2 - dz^2)}{dz^2} = \frac{\mathrm{D}\,.\,Y\,.\,dy\,.\,dx}{a},$$

or, $-\dfrac{d^2z}{dz^2}\,.\,dx = \dfrac{\mathrm{D}\,.\,Y\,.\,dy}{a}$, and integrating

$$\frac{dx}{dz} = \frac{Y \pm c}{a}, \text{ since } Y = \int \mathrm{D}\,.\,Y\,.\,dy = \int \frac{dY}{dy}\,dy =$$

$\int dY$, and $\dfrac{c}{a}$ is the correction; hence,

$$\frac{dx^2}{dx^2 + dy^2} = \frac{(Y \pm c)^2}{a^2}, \text{ and}$$

$$dx = \frac{(Y \pm c)\,.\,dy}{\sqrt{(a^2 - [Y \pm c]^2)}}.$$

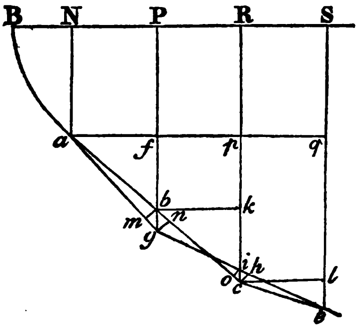
If $Y = f(y) = y$, then $\mathrm{D}\,.\,Y = \dfrac{dY}{dy} = \dfrac{dy}{dy} = 1$, and

$$\int dx = \int \frac{(y \pm c)\,.\,dy}{\sqrt{(a^2 - [y \pm c]^2)}},$$

$$\text{or } x = C - \sqrt{(a^2 - [y \pm c]^2)}$$

an equation to a circle. Hence, of all curves, of a given length, that can be drawn between two points, the circle is the curve that contains the greatest area.

If we apply Bernoulli's fundamental equations to the second case of the Isoperimetrical problem, [see p. 13.]



that is, when the ordinates are not functions of the ordinates Pb, Rc, &c., but of the arcs Bab, $Babc$, &c., the process will be somewhat different from the preceding. For then we have, instead of

$$f(Pb)+f(Rc)=f(Pg)+f(Ri)$$

$$f(Bab)+f(Babc)=f(Bag)+f(Bagi)$$

or, $f(Bab)-f(Bab+gm)=f(Babc-co)-f(Babc)$

and, as before, $\text{D}.f(Bab).gm=\text{D}.f(Babc).co$

But the fundamental equation involves bg, ci; therefore, gm, co, must be expressed in terms of them. Now, by similar triangles,

$$gm=\frac{fb}{ab}.bg,\quad co=\frac{le}{ce}.ci;\ \text{hence}$$

$$\text{D}.f(Bab):\frac{fb}{ab}.bg=\text{D}.f(Babc).\frac{le}{ce}.ci;$$

and substituting in the fundamental equation [p. 15. l. 9.]

$$\left(\frac{fb}{ab}-\frac{kc}{bc}\right).\frac{ab}{fb.\text{D}.f(Bab)}=\left(\frac{kc}{bc}-\frac{le}{ce}\right).\frac{ce}{le.\text{D}.f(Babc)}.$$

Now, Bernoulli remarks, that in this equation, as it stands, there is not that uniformity, which enables us to pass from one element ab to another bc affected in the same manner, and so on. There would have been the requisite uniformity, if the factor in the right hand side of the equation, instead of $\frac{ce}{le}$, had been $\frac{bc}{kc}$. We shall cause therefore the requisite uniformity, if we multiply each side by $\frac{bc}{kc}$, and there results the *specific* * equation,

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* See Note, p. 19.

$$\left(\frac{fb}{ab} - \frac{kc}{bc}\right) \cdot \frac{ab}{fb} \cdot \frac{bc}{kc} \cdot \frac{1}{\mathrm{D} f'(Bab)} = \left(\frac{kc}{bc} - \frac{le}{ce}\right) \cdot \frac{bc}{kc} \cdot$$

$\frac{ce}{le} \cdot \frac{1}{\mathrm{D}.f'(Babc)}$ = a constant quantity; hence, since $\frac{ab}{fb} = \frac{bc}{kc}$, and calling $\mathrm{D}.f'(Bab) = S$, we have,

$$-d.\left(\frac{dy}{dz}\right) \cdot \frac{dz^2}{dy^2 . S} = \frac{dx}{a},$$

or $\frac{d^2z.dy - d^2y.dz}{dz^2} = \frac{S}{a} \cdot \frac{dy^2}{dz^2} . dx$; but, $d^2y.dy = d^2z.dz$;

$$\therefore -\frac{d^2y.dx^2}{dz^3} = \frac{S}{a} \cdot \frac{dy^2}{dz^2} . dx, \text{ or } -\frac{d^2y}{dy^2} . dx = \frac{S.dz}{a};$$

Hence, $\frac{dx}{dy} = \frac{1}{a} (\int S.dz + c) = \frac{Z+c}{a}$, if $Z = \int S.dz$,

or, $a.dx = dy\ (Z + c)$, the equation to the curve.

This, as we have before remarked, is, in the history of Isoperimetrical Problems and their connected Calculus, the most important problem; since, as long as John Bernoulli made two elements of the curve only to vary, he constantly, by his erroneous solutions, afforded matter of triumph to his brother James. The above solution is John Bernoulli's last solution, and an exact one; but, it was not published till 1718, in the Memoirs of the Academy of Paris for that year, sixteen years after the appearance of James Bernoulli's *.

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* This solution appeared in the *Acta Erud. Lips.* Mai, 1701, p. 213. John Bernoulli's solution (which was faulty) was communicated to Leibnitz in 1698, to the Academy of Sciences in 1701, and published in their Memoirs for 1706.

On this solution, that of John Bernoulli's is essentially founded; and indeed on the very principle, the neglect of which had vitiated all his former solutions; containing also, under the form of fundamental equations*, James Bernoulli's proportions †. The latter are, with the slightest trouble, transmutable into the former. Yet John Bernoulli, with a total disregard of justice, wishes to confer superiority on his method, by the reverse operation; that is, by transforming the former into the latter. This, however, is not the sole nor the least trait of his unfairness.‡

John Bernoulli's solution possesses greater elegance and compactness than his brother's. Yet these are qualities which we may attribute, almost with equal justice, as well to time as to genius. It possesses, however, a characteristic excellence in the uniformity of its *specific* equations. A considerable advancement was thereby made in the calculus of Isoperimetrical problems; and it is on a like principle of uniformity, that certain methods of solution, of more recent date, are founded*.

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* *Mem. Acad.* 1718, p. 103, &c.

† *Acta Erud. Lips.* 1701, p. 213, or James Bernoulli's Works, tom. II. p. 899, or John Bernoulli's Works, tom. II. p. 220, &c.

‡ The palpable error of his former solution he wishes to reduce to a mere fault of inadvertence. "Pour reparer cette faute d'inadvertence," he says, "Je vas donner ici une nouvelle manière de resoudre, &c." Besides, he is ever seeking occasion of aspersing his brother's method, [see *Mem. Acad.* 1718, pp. 102, 103, 131.] and this he does sixteen years after James Bernoulli's death: when, that event, the lapse of time, the recollection of his brother's kindness, a zeal for a brother's fame, ought to have assuaged and laid to sleep all angry passions.

Some of the latter assertions the Student must be content to take on trust: for their proof would lead into too wide a digression. Yet, his curiosity will be amply rewarded, if he will search for, what may be found, the identity of the methods employed by James Bernoulli in his "Solutio magni Problematis Isoperimetrici," and by his brother in his last essay, inserted in the Memoirs of the Academy of Sciences.

With the above-mentioned methods, the researches of the Bernoullis on these subjects terminated. Towards the period of their close, in 1715, Brook Taylor, in his "Methodus Incrementorum," solved the problem of the Isoperimetricals, on principles not different from those of the Bernoullis, but with some alteration of symbolical notation. The most material alteration, or rather improvement, consisted in representing the fluxion of *V*

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* In Lyons's *Fluxions* (1758), at page 99, the author, in solving the *Brachystochrone*, arrives at this equation,

$$\frac{\dot{y}}{V\sqrt{(\dot{x}^2+\dot{y}^2)}} = \frac{\dot{w}}{v\sqrt{(\dot{x}^2+\dot{w}^2)}};$$

in which $\dot{y}$, $\dot{w}$, are the fluxions of two contiguous ordinates, and V, v, the velocities in the arcs. Hence, since there is in this equation a law of uniformity, precisely of the same nature as that which Bernoulli pointed out; the author infers, that $\frac{\dot{y}}{V\dot{s}}$ ($\dot{s}$ the fluxion of the arc) is a given quantity.

The same principle of uniformity serves also, in part, as the basis of Mr. Vince's solutions, *Fluxions*, p. 195, and of some of Emerson's, pp. 182, 183, &c. Third edition.

when $\int V\dot{x}$ is the analytical expression of the maximum property; thus,

$$\dot{V} = M\dot{x} + N\dot{y} + L\dot{s}$$

which mode of expression, Euler, as we shall hereafter see, skilfully availed himself of.

The methods of the Bernoullis and of Taylor, were held, at the time of their invention, to be most complete and exact. Several imperfections, however, belong to them. They do not apply to problems involving three or more properties; nor do they extend to cases involving differentials of a higher order than the first: for instance, they will not solve the problem, in which a curve is required, that, with its radius of curvature and evolute shall contain the least area. Secondly, they do not extend to cases, in which the analytical expression contains, besides x, y, and their differentials, integral expressions; for instance, they will not solve the second case proposed in James Bernoulli's Programma [see p. 12.] if the Isoperimetrical condition be excluded; for then the arc s, an integral, since it $= \int dx \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}$, is not given. Thirdly, they do not extend to cases, in which the differential function expressing the maximum should depend on a quantity, not given except under the form of a differential equation, and that not integrable; for instance, they will not solve the case of the curve of the quickest descent, in a resisting medium, the descending body being solicited by any forces whatever.

There are no other peculiarities in Taylor's method, than those we have mentioned, that demand our attention. It has no recommendation from its neatness and perspicuity, but is justly censured by John Bernoulli for its obscure conciseness *.

We must now direct our attention to a period of greater interest, during which, the great Euler, who left no part of science untouched or unadorned, directed his attention to the calculus of Isoperimetrical problems.

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* "M. Taylor, homme d'esprit, et Geometre tres habile, qui a heureusement pénétré jusque dans ce que nous avons de plus profond, comme il parôit par son livre de Methodo Incrementorum: sentant bien la longueur embarassante de l'analyse de mon frère, et voulant la rendre plus courte et un peu claire, a repandu lui-même tant d'obscurité sur cette matiere (aussi bien sur d'autres où il a voulu etre court) qu'il semble y prendre plaisir, et que je doute qu'il y ait quelqu'un, quelque pènètrant qu'il soit, qui l'entende partout, quand même la matiere lui seroit dèja connue d'ailleurs," &c. *Mem. Acad.* 1718, p. 103.

CHAP. III.

Euler's first Memoir on Isoperimetrical Problems — Table of Formulæ — Solution of Problems by it — Methods of Thomas Simpson, Emerson, and Mac Laurin.

An interval of fifteen years had elapsed, before the subject of Isoperimetrical problems was resumed by Euler. He took it up where the Bernoullis had left it, and conducted his first investigations on their plan. These are not easily intelligible without a previous knowledge of the researches of his great predecessors. The Student, who, for initiation into the peculiar calculus of Isoperimetrical problems, should resort to Euler's first memoir, would, from the novelty of its terms, principles, and methods, find it perplexed and abrupt. But, he will fancy himself gliding along the same route, if after Bernoulli, he takes Euler as the guide of his enquiries*.

Euler's first memoir was inserted in vol. VI. of the Ancient Commentaries of Petersburg for 1733. He there distributes his problems into classes. In the first, are problems like that of the *brachystochrone* and the curve of least resistance, with the property of the minimum, but without the Isoperimetrical condition or any other. These are to be solved from the principle of the property of a maximum, belonging to the elements of the curve, as well as to the curve itself; and from the principle of the equality between two proximate states of a quantity, when near its minimum or maximum *; and they require for their solution the variation of two elements only of the curve. Thus, in the subjoined Figure which is constructed similarly to Euler's †, let *CLND* be the element that possesses the property of a proposed maximum or minimum: take *CMGD* indefinitely near; then Euler says, if we express the property in *CMGD* and in *CLND*, the difference of the two expressions ought to equal o.

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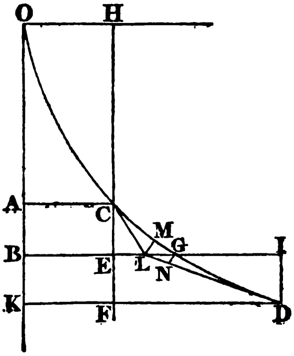
* It is thus with other subjects. Investigation is easy when it is made with the proper series of steps; but difficult, when by our own sagacity and labour, we must supply either steps that are wanting, or approximate those which are too widely separate.



[a] Now CL is changed into $CG = CM + MG = CL + \frac{LE}{CL} . LG$,

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* See p. 7. † *Comm. Acad. Petrop.* tom. VI. p. 128.

. [b] and LD is changed into $DG = DL - LN = DL - \frac{LI}{LD}.LG$,

or the variations of CL, LD are $\frac{LE}{CL}.LG$, $-\frac{LI}{DL}.LG$.

Hence, in an instance, in which $\int x^n.ds$, is to be a minimum (x = abscissa, ds = differential of the curve,) we have,

$$OA^n.CL + OB^n.LD = OA^n.CG + OB^n.GD,$$

hence, by [a], [b],

$$OA^n.\frac{LE}{CL}.LG = OB^n.\frac{LI}{LD}.LG,$$

$$\text{or } OA^n.\frac{LE}{CL} = OB^n.\frac{LI}{LD},$$

and consequently from the *uniformity* of this equation, [p. 15, 20, &c.]

$OA^n.\frac{LE}{CL}$, or $x^n.\frac{dy}{ds} = a$ the equation to the curve.

If $n = -\frac{1}{2}$, $\frac{dy}{ds\sqrt{x}} = a$, the equation to a cycloid, and the expression for the minimum is $\int \frac{ds}{\sqrt{x}}$, or $\int dt$ (t the time,) or the time is a minimum down the curve, whose equation is $\frac{dy}{ds\sqrt{x}} = a$. This problem is that which was solved in pp. 4, 9. and on the same principles; for if we compare the solutions, the two latter will be found to differ from the former, merely in the greater compactness and regularity of their processes, and in their furnishing something like a clue to the general solution of similar problems*.

The preceding problem is easily generalised; for if P be a function of x, and $\int Pds$ is to be a minimum, we should have, supposing P to become P' when x becomes $x+dx$,

$$P.CL + P'.LD = P.CG + P'.GD$$

and by [a], [b],

$$P.\frac{LE}{CL}.LG = P'.\frac{LI}{LD}.LG, \text{ or } P.\frac{LE}{CL} = P'.\frac{LI}{LD}$$

an equation, which, according to Bernoulli, may be called a specific equation.

Hence, $P.\frac{dy}{ds} = a$, the equation to the curve.

By these methods, hardly to be reckoned different from those of the Bernoullis, Euler solves problems of the first class: he then passes on to those of the second, which besides the property [B] of the maximum, are to have another property [A] †; of this class are those problems solved by James and John Bernoulli, in which the property [*A*], is the Isoperimetrical one. These require the variation of three elements *ab*, *bc*, *ce*;

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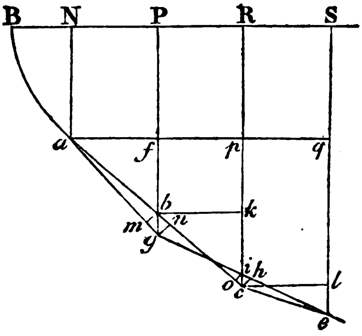
* The first solution of James Bernoulli, as it stands [see p. 4.] affords scarcely the least glimpse of a general method or formula of solution. Of its peculiarity, the Editor of John Bernoulli's Works is aware, since he remarks, "Quoniam autem synthesin meram particularem, nec ad similes casus facile applicandam, &c."

† The following is Euler's classification:
"I. Ex omnibus prorsus curvis eam determinare, quæ proprietatem *A* maximo vel minimo gradu contineat. II. Ex omnibus curvis proprietate *A* æqualiter præditis, eam determinare, quæ proprietatem *B* maximo



and from such John Bernoulli procures, in fact, two equations, involving *bg*, *ci*: one equation * called a fundamental equation, from the Isoperimetrical property [*A*]; the other † from the property of the maximum [*B*]: and these equations may be thus represented:

[*c*] . . . $P.bg - Q.ci = 0$ from Isop^l. condit. [*A*],
[*d*] . . . $R.bg - S.ci = 0$ from max^m. condit. [*B*];

whence results $PS = QR$, which, if the law of uni-

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maximo vel minimo gradu contineat. III. Ex omnibus curvis et proprietate *A* et proprietate *B* æqualiter præditis eam determinare, quæ proprietatem *C* maximo minimove gradu contineat. IV. Ex omnibus curvis proprietatibus *A* et *B* et *C* singulis æqualiter præditis, eam determinare, quæ proprietatem *D* maximo minimove gradu contineat.——Simili modo quinta classis curvas quatuor proprietatibus præditas contemplabitur et ita porro sequentes." *Comm. Acad. Petrop.* tom. VI. p. 125.

[*] p. 15. [†] p. 19.

formity prevailed in it, as in the two first instances [*], would be a specific equation: but, if the necessary uniformity should be wanting, then it must be introduced, by multiplying each side of the equation by proper quantities, as was done in the solution of the third problem [†]. Now Euler's plan and reasoning is very similar to this: he deduces the equations [*c*] [*d*]; and then observes, that the quantities Q and S are frequently so compounded, that $Q = P + dP$, and $S = R + dR$; or, if not so compounded, may, by multiplication, be reduced to that form [‡].

Euler, in this part, gives no general proof of the preceding assertion; but, if it be admitted, then the equation

$$QR = PS, \quad \text{becomes}$$

$$R(P+dP) = P(R+dR), \quad \text{and consequently,}$$

$$R.dP = P.dR, \text{ or } \frac{dR}{R} = \frac{dP}{P}, \quad \text{and integrating}$$

$$P + aR = 0, \quad \text{the equation to the curve.}$$

In point of practical convenience of solution, a great step was made by Euler in the preceding process. For the computist is directed to attend solely to the deriving, from one of the proposed properties, an equation of the form,

$$P.bg - (P+dP)\,ci;$$

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[*] pp. 20, 24. [†] p. 26.

[‡] "In quibus quantitates $\mathfrak{Q}$ et S plerumque ita sunt comparatæ, ut sit $\mathfrak{Q} = P + dP$, et $S = R + dR$. Si vero hujusmodi formam non habuerint, poterunt semper multiplicando vel dividendo æquationes ad talem reduci." *Comm. Acad. Petrop.* tom. VI. p. 134.

whence P will be known. He must then derive from the other of the properties a similar equation,

$$R.bg - (R + dR).ci$$

whence R will be known: and the resulting equation of solution will be

$$P + aR.$$

For instance, suppose one of the properties [B], to be such, that $\int y^n.dx$ shall be a maximum; then, since dx is constant,

$$Na^n + Pb^n + Rc^n = Na^n + Pg^n + Ri^n$$

and $\therefore\ Pb^n - (Pb + bg)^n = (Rc - ci)^n - Rc^n$,

and, $Pb^{n-1}bg = Rc^{n-1}.ci$,

or $Pb^{n-1}.bg - Rc^{n-1}.ci = 0$,

which, since Rc is the contiguous or consecutive ordinate to Pb, and therefore (if $Pb = y$) $= y + dy$, is of the proposed form,

$$P.bg - (P + dP)\,ci,$$

in which $P = y^{n-1} = \dfrac{d.y^n}{dy}$, or to render the equation homogeneous, $y^{n-1}.dx$; similarly, if T had been a function of y and the property $\int T.dx$; P would have been $= \dfrac{dT}{dy}$ or $\dfrac{dT}{dy}.dx$. Suppose the other property [A] to have been $\int x^m.ds = a$, a being constant and ds the element of the curve, then

$BN^m.ab + BP^m.bc + BR^m.ce = BN^m.ag + BP^m.gi + BR^m.ie$,

and substituting according to the forms [a], [b], p. 33. there will result

$\left(BN^m\frac{bf}{ab} - BP^m.\frac{kc}{bc}\right).bg - \left(BP^m.\frac{kc}{bc} - BR^m\frac{le}{ce}\right).ci$ $= 0$ *, which is evidently under the proposed form, if we make $R = BN^m.\frac{bf}{ab} - BP^m\frac{kc}{bc}$, or symbolically $=$ $d\left(\frac{x^m.dy}{ds}\right)$. Similarly, if instead of $\int x^m\, ds$, the form had been $\int X.ds$, X a function of x, then there would have resulted,

$$R = d\left(\frac{X.dy}{ds}\right).$$

The equation therefore to the curve with these two properties, A and B,

$$\text{is } y^{n-1}.dx + a.d\left(\frac{x^m.dy}{ds}\right)$$

$$\text{or generally, } \frac{dT}{dy}\,dx + a.d\left(X.\frac{dy}{ds}\right).$$

Since the equation $P.bg - (P + dP)\ ci$, derived from the property B, is precisely similar to the equation, $R.bg - (R + dR)\ ci$, derived from the property A; it is plain, that the resulting equation, that which determines the curve, or the relation between y and x, will be the same, if B and A were changed; that is, if

$$P.bg - (P + dP)\ ci$$

be derived from A, and

$$R.bg - (R + dR)\ ci$$

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* This is exactly under the form of John Bernoulli's *specific* equations, for the law of uniformity is manifest in it.

from *B*. For instance, suppose the curve to be required which, with a given length, should contain the greatest area; here the maximum property *B* is $\int y\,dx$, and the Isoperimetrical property *A*, is $\int\sqrt{(dx^2+dy^2)}=a$. The resulting equation is $dy=\frac{x\,dx}{\sqrt{(r^2-x^2)}}$ an equation to a circle: which equation would equally result, if we required the curve which, with a *given area*, should contain the *greatest arc*.

This important remark of the commutability of the properties was originally made by James Bernoulli *.

Euler, we have seen, reduced problems of the second class to a dependance on two similar equations of the form

$$P.bg-(P+dP)\,ci=0,$$

the determination of *P* depending on the proposed properties: for instance, if the property *A* or *B* were $\int T.dx$, $T=f(y)$, *P* would equal $\frac{dT}{dy}$, or $\frac{dT}{dy}.dx$. If the property were $\int T.dy$, $T=f(x)$, P would $=\frac{dT}{dx}$ or $\frac{dT}{dx}.dx$. If the property were $\int T.ds$, $T=f(x)$, P would equal $d\left(T.\frac{dy}{ds}\right)$; and by observation on the resulting forms for P, Euler generalised his conclusions, and arranged them in a table, after the manner of the subjoined specimen[*]:

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* "In quibus omnibus singularis quædam observatur reciprocatio. Quemadmodum enim, exempli gratiâ, inter omnes figuras ejusdem Perimetri circulus maximam possidet aream, catenaria maximam conversione sui gignit superficiem, solidumque maximum elastica; sic inter omnes vicissim figuras quæ aut æqualibus gaudent areis, aut æquales rotatione gignunt superficies, solidave æqualia; circulus, catenaria et elastica minimo clauduntur ambitu, quod pariter procedit in omnibus aliis." *Acta Erud. Lips.* Mai, 1701, p. 213. or *Opera*, tom. II. p. 919. or, John Bernoulli, tom. II. p. 234.—Euler makes the same remark. *Comm. Acad. Petrop.* tom. VI. pp. 135, 150. and tom. VIII. p. 175. and in his *Methodus Inveniendi Curvas, &c.* See also Emerson's *Fluxions*, p. 170.

Proprietates propositæ.		Valores Litteræ P respondentes.
I. $\int T.dx$	$dT=Mdy$[†]	$P=M.dx$
II. $\int T.dy$	$dT=Ndx$	$P=N.dx$
III. $\int T.ds$	$dT=Ndx$	$P=d.\left(T.\frac{dy}{ds}\right)$
IV. $\int T.ds$	$dT=Mdy$	$P=d.\left(T\frac{dy}{ds}\right)-Mds$,
&c.		

and of these forms he gave fifteen, by reference to which, any problem belonging to the second class might be solved.

For instance, suppose the curve to be required, which, amongst all others of the same length, should contain the greatest area. Here, the maximum property B, $=\int y\,dx$,

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[*] *Comm. Acad. Petrop.* tom. VI. p. 141.

[†] $dT=Mdy$, and $M=\frac{dT}{dy}$, or M is the differential coefficient of T, making in T, y to vary. Similarly, $N=\frac{dT}{dx}$ is the differential coefficient making in T, x to vary. If T should contain both x and y, that is, if $dT=Mdy+Ndx$, then M and N would become partial differential coefficients. See *Princ. Anal. Calc.* p. 79.

the Isoperimetrical A, $=\int ds = a$.

By Form I. $T=y$; $\therefore M=1$, $P=dx$.

By Form III. $T=1$; $P=d\left(\frac{dy}{ds}\right)$ or $R=d\left(\frac{dy}{ds}\right)$

Hence, since the equation is $P \pm aR=0$ [see p. 38.] $dx \pm a.d\left(\frac{dy}{ds}\right)=0$, or $x+a\frac{dy}{ds}-c=0$ (c = correction); and by reduction,

$$dy=\frac{(x-c).dx}{\sqrt{(a^2-[x-c]^2)}}$$ an equation to a circle *.

Again, suppose the curve to be required, which, amongst all others of the same length, shall, by a rotation round its axis, generate the greatest solid. Here, $B=\int y^2.dx$; $\therefore$ by Form I. $T=y^2$, $M=2y$, $P=2y.dx$,

$A=\int ds$; $\therefore$ by Form III. P or $R=d\left(\frac{dy}{ds}\right)$;

Hence, $2y.dx+a.d\left(\frac{dy}{ds}\right)=0$,

$$\text{or, } 2y.dx+a.\frac{ds.d^2y-dy.d^2s}{ds^2}=0.$$

But since dx is constant, and $ds^2=dx^2+dy^2$, $ds.d^2s=dy.d^2y$; therefore, substituting,

$$2y.dx+a.\frac{d^2y.dx^2}{ds^3}=0,$$

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[*] See Emerson's *Fluxions*, third edition, p. 187; also Simpson's *Fluxions*, p. 485.

and, $2y + \frac{a.d^2y.dx}{(dx^2+dy^2)^{\frac{3}{2}}} = 0$; multiply by dy, and integrate, and we have

$$y^2 - \frac{a.dx}{(dx^2+dy^2)^{\frac{1}{2}}} = c \quad [c = \text{correction,}]$$

whence, $dx = \frac{(y^2-c)\,dy}{\sqrt{(a^2-[y^2-c]^2)}}$, an equation to the elastic curve; and which in a particular case, when $c=0$, becomes

$$dx = \frac{y^2.dy}{\sqrt{(a^2-y^4)}},$$

and the curve in this case is called the rectangular elastic curve *.

As a third example, let the curve be required, which, amongst all others of the same length, shall have its center of gravity most remote from the axis. Here, (calling x the distance from the axis) $B = \int\frac{x.ds}{s}$; ∴ by Form III. [since s is a given quantity] $P = d\left(x\frac{dy}{ds}\right)$ again $A = \int ds$; ∴ by Form III. P, or $R = d\left(\frac{dy}{ds}\right)$

$$\text{Hence, } a.d\left(\frac{dy}{ds}\right) + d\left(x\frac{dy}{ds}\right) = 0;$$

$$\therefore\ (a+x)\frac{dy}{ds} = c, \text{ or } c\,ds = (a+x).dy$$

an equation to the catenary.

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* See Simpson's *Fluxions*, p. 486, where the solution is not general.

This example could not have been solved by Euler's table, if the property had been any other than the Isoperimetrical: for s, an integral, $= \int dx \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}$; and Euler gives, in this memoir, no general method of finding the resulting equation, such as P is in his table, when the analytical expression of a property involves integrals. See tom. VI, p. 144.

By means of this table, the practical solution of Isoperimetrical problems, was, as it has been already said, very materially expedited. In a subsequent part of his memoir*, Euler increases his table by nine new forms: making the whole number twenty-four. And although this table is now superseded, yet its examination is not without interest, since we may discover in it the parcels of that general formula, which the author afterwards exhibited.

Having given rules of solution for all problems belonging to the second class, Euler passes on to those of the third. In this a curve is required, which, amongst all others equally possessing the property A and the property B, contains a property C of maximum or minimum. For instance, if the curve be required, which, amongst all curves of the same length and the same area, is such, that the time down it is a minimum. This class requires the variation of four elements of the curve; and consequently the methods of the Bernoullis, without extension, are insufficient. The course pursued by Euler is like his former one in the second class. He constructs a diagram similar to the one, p. 36, the sole difference arising from the introduction of a fourth element of the curve; and he reduces the analytical solution to a dependence on three similar equations instead of two.

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* *Comm. Petrop.* tom. VI, p. 146.

The general form of these equations is,

$$P.bg - Q.ci + R.d\delta$$

$d\delta$* being a variation of the ordinate similar to the variations bg and ci; and he makes an observation similar to the one in p. 37; namely, that it frequently happens in simple cases, that $Q = P + dP$, $R = P + 2dP + d^2P$; but, if such should not be the form of Q and R, then the skill of the analyst must be directed to reduce them under that form.

When three equations, such as

$$P.bg - (P + dP)\,ci + (P + 2dP + d^2P).d\delta = 0$$
$$p.bg - (p + dp)\,ci + (p + 2dp + d^2p).d\delta = 0$$
$$\pi.bg - (\pi + d\pi)\,ci + (\pi + 2d\pi + d^2\pi).d\delta = 0$$

are obtained, the resulting equation to the curve will be†

$$P + ap + b\pi = 0 \qquad [1]$$

for, taking the differential, $dP + a.dp + b.d\pi = 0 \qquad [2]$

and again, $d^2P + a.a^2p + b.d^2\pi = 0 \qquad [3]$

And since, $[1] \times bg = 0$; $([1] + [2]) \times ci = 0$, and $([1] + 2[2] + [3]) \times d\delta = 0$;

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* $d\delta$ has no connexion whatever with the separate symbols d, δ

† Euler, *Comm. Acad. Petrop.* tom. VI, p. 149.

$$\begin{array}{l} \therefore\ P.bg \quad + \quad ap.bg \quad + \quad b\pi.bg = 0 \\ -(P + dP).ci - a(p + dp).ci - b(\pi + d\pi).ci = 0 \\ (P + 2dP + d^2P).d\delta + a(p + 2dp + d^2p)d\delta + b(\pi + \\ 2d\pi + d^2\pi)\,d\delta = 0; \end{array}$$

which equations, adding the quantities that are placed vertically, verify themselves; and accordingly shew, that the equation

$$P + ap + b\pi = 0$$

satisfies the three equations.

The solution of problems then would be similar to that of those in the second class, and would be immediately obtained, if the quantities P, p, π, were contained in his table of forms *; and in many cases they are. For instance, suppose the curve to be required, which amongst all other curves, of the same length and same area, generates, by a rotation round an axis parallel to the ordinate y, the greatest solid. Here,

the max^{m}. property C is $\int x^2 dy$; $\therefore$ by II. p. 41. $P = 2xdx$,
the . . . property A is $\int y dx$; $\therefore$ by I. $p = dx$,
the . . . property B is $\int ds$; $\therefore$ by III. $\pi = d\left(\frac{dy}{ds}\right)$;

and consequently, the equation to the curve is

$$2xdx + a.dx + b.d\left(\frac{dy}{ds}\right) = 0;$$

and integrating, $x^2 + ax + b\frac{dy}{ds} = c$ [c = correction]

$$\text{whence } dy = \frac{(x^2 + ax - c)\,dx}{\sqrt{(b^2 - [x^2 + ax - c]^2)}}.$$

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* *Comm. Petrop.* tom. VI, p. 151.

Again, as a second instance, let us take a third case of the *brachystochrone*, of which we have already had two, pp. 4, 18; that is, besides the condition C of quickest descent, let the conditions A and B be those of a given length, and a given area;

$$\begin{aligned} \text{here, } C &= \int \frac{ds}{\sqrt{x}}; & \therefore \text{ by III. } P &= d\left(\frac{dy}{ds\sqrt{x}}\right) \\ A &= \int y\,dx; & \therefore \text{ by I. } p &= dx \\ B &= \int ds; & \therefore \text{ by III. } \pi &= d\left(\frac{dy}{ds}\right) \end{aligned}$$

and consequently, the equation to the curve is

$$d\left(\frac{dy}{ds\sqrt{x}}\right) + a\,dx + b.d\left(\frac{dy}{ds}\right) = 0;$$

and integrating, $\dfrac{dy}{ds\sqrt{x}} + ax + b\dfrac{dy}{ds} = c.$

Euler, as we have observed, commenced his researches where the Bernoullis had terminated theirs; but, at the end of this his first investigation, he was considerably removed from the original point of departure. Several important objects had been attained by him; the solution of problems involving three or more properties; the reduction of such problems to a dependence on two or more similar equations; the solution of problems of the first class, and of some of higher classes, by a more general method *, and by reference to a table of formulæ.

These methods were held, by their author, to be so complete, that, on the ground of facility, nothing farther was to be desired*. They are, however, not exempt from several imperfections. Problems, involving the differentials of x or y of an higher order than the second cannot be solved by them: for instance, that which requires, amongst all other curves, one, in which $\int\frac{d^3y}{dx.dy}$ is a maximum or minimum. Secondly, problems cannot be generally solved by them which involve integrals, such not being constant: for instance, that in which it should be required to find a curve that amongst all other curves, has its center of gravity lowest. Euler solves this problem when another condition, that of the Isoperimetrical property, is added: for then the arc s, the integral of $dx\surd\left(1+\frac{dy^2}{dx^2}\right)$, is in all curves supposed to be the same †.

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* Tamen eas methodo paulisper diversa et multo latiùs patente sum persecuturus, &c. *Comm. Petrop.* tom. VI. p. 127.

The researches of Maclaurin ‡, Emerson §, and Simpson ||, on this subject, may here be noticed. With regard to practical methods of solution they do not extend so far as those of Euler, which we have been speaking of; and in point of perspicuity, if we except Maclaurin, the other two mathematicians are inferior to the learned foreigner.

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[*] Atque methodo tam facili solutum ut nihil amplius desiderari posse videatur. *Comm. Petrop.* tom. VIII, p. 159.

[†] See p. 43.

[‡] *Fluxions*, p. 478. [§] *Fluxions*, Third edition, p. 170.

[||] *Fluxions*, vol. II. p. 480. Third edition, 1750.

The methods of Maclaurin and Simpson (for Emerson's is plainly taken from that of the former) extend to cases in which more than one property is involved: but they are inapplicable to the three cases, and the connected problems enumerated in p. 30.

Maclaurin's formula of solution is this: If X and Z are functions of x, then if $Xds - Zdy$ be a minimum or maximum, $Xdy = Zds$. This result is included amongst Euler's. [See *Comm. Acad. Petrop.* tom. VI. pp. 141, 142, 143.] For since Xds expresses one property, and $dX = \frac{dX}{dx}dx$, or since X is a function of x, we have, by Form III. the quantity corresponding to P [see p. 41.] $= d\left(X.\frac{dy}{ds}\right)$, and for Zdy expressing the other property, by Form II. the quantity, corresponding to $P, = \frac{dZ}{dx}.dx$; consequently, the resulting equation is

$$d\left(X\frac{dy}{ds}\right) = a.dZ \quad \text{and} \quad Xdy = aZds,$$

the same result in fact as Maclaurin's.

Simpson's method is equally restricted with Maclaurin's: it rests too on the assumption of the principle, that the property of minimum or maximum, true for the whole curve, is true also for any portion of it. The want of generality, therefore, in this principle, would vitiate the method in its application to the excepted cases.

The methods just described solve not problems of greater depth and intricacy than those of the Bernoullis; although, it must be remarked, they are invested with greater analytical neatness and compactness. They are not, however, more perspicuous; and, even if they did possess greater extent and clearness, it would not suit the purpose of the present Tract, longer to insist on them, since they conduct us not towards that formula and algorithm, with which the researches on this subject have been closed.

The plan of this Work now leads us to the consideration of Euler's second memoir; in which, he very materially improved the calculus of Isoperimetrical problems; although he introduces his resumed researches with a prefatory remark, that his former methods of solution, on the footing of facility, left nothing farther to be desired *.

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* Atque methodo tam facili solutum, ut nihil amplius desiderari posse videatur. *Comm. Petrop.* tom. VIII, p. 159.

CHAP. IV.

Euler's second Memoir—General Formulæ of Solution—Characters of distinction, which Problems admit of—Exceptions to the general Formulæ.

The formation of the table * in Euler's former memoir depended on this principle: If $\int V dx$ is the analytical expression of the maximum property †, certain functions of x, y, &c. are substituted for V, and thence a resulting equation (according to Euler the equation $P = 0$), is obtained. In the present memoir, he represents the differential of V, or dV, as Taylor had done [p. 30.]; that is, he puts

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* P. 41.

† Since in the succeeding part of this Tract $\int V dx$ will be frequently used as the analytical expression of the maximum property, it may not be improper to illustrate its meaning by one or two examples.

In the *brachystochrone*, the time $= \int \frac{ds}{\sqrt{y}} =$ a minimum; therefore $\int \frac{ds}{\sqrt{y}} = \int \frac{\sqrt{(dx^2 + dy^2)}}{\sqrt{y}} = \int \frac{dx\sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}{\sqrt{y}}$, which, compared with $\int V dx$, gives $V = \frac{\sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}{\sqrt{y}} = \frac{\sqrt{(1+p^2)}}{\sqrt{y}}$ $\left[p = \frac{dy}{dx}\right]$.

In the solid of least resistance, the resistance $= \int \frac{y . dy^3}{dx^2 + dy^2} = \int \frac{y . p^3}{1+p^2} . dx$; therefore, comparing, $V = \frac{yp^3}{1+p^2}$.

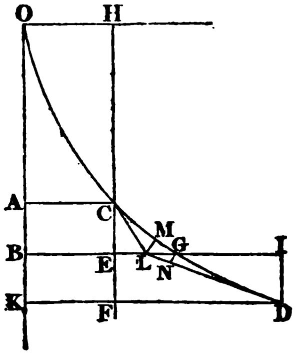
If, with a given area, the curve generating the solid of least surface is required, then [see p. 37.] $\int V dx = \int (2\pi y . ds + ay dx) = \int (2\pi y \sqrt{[1+p^2]} . dx + ay\, dx)$; $\therefore V = ay + 2\pi y \sqrt{(1+p^2)}$.

And from these expressions, the value of dV may be found by the common rules of the differential or fluxionary calculus.

$$dV = Mdx + Ndy + Pdp + \&c. + Lds\ ^{*}$$

$$\left[p = \frac{dy}{dx},\ \text{and}\ s = \text{arc,}\right]$$

and from this more general formula of representation, he deduces an equation comprehending almost all his former particular equations which he had registered in a table [p. 41.]. His mode of proceeding is not materially different from his former mode. If V contains no integrals, but is merely a function of x, y, dy, &c., then



the property expressed by $\int V dx$ is common to the curve, and to its elementary portions: hence, if we suppose two portions only of the curve to vary, and if $V dx$ corresponds to CL, and $V'.dx$ to LD, $(V + V')$

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[*] P in this and the following pages is merely a coefficient, and different from the P of the preceding Chapter.

dx is a minimum. Hence, that peculiar differential of $(V+V')\,dx$, which arises from changing CL, LD, into CG, GD, is equal to nothing: which differential is easily obtained from the common differential, since each has this in common, that of being the first term of the difference of two successive values: now the common differential,

$$\text{or, } (dV+dV')\,dx =$$
$$(Mdx+Ndy+Pdp)\,.\,dx+(M'\,dx+N'dy'+P'dp')\,dx,$$

supposing the differential of V to consist of three terms only, and M', N', P', to be the values of M, N, P, when V becomes V'. Now, in order to compute the changes in the several terms, we may observe, that since $AC=y$, and since dx is supposed to be constant, the terms Mdx, Ndy, are not affected by the translation of the point L to G; but Pdp is; for $p=\frac{dy}{dx}=\frac{EL}{CE}$: but when L is transferred to G, $p=\frac{EG}{CE}=\frac{EL+LG}{CE}$; consequently the variation produced in p, or what dp becomes, when it expresses the peculiar differential which we are seeking, is $\frac{LG}{CE}$*. Hence the term Pdp

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* Those who are possessed of the calculus of variations will here perceive the very great advantage accruing from the mere invention of a symbol δ to denote an operation, bordering on, yet distinguished from, the operation which d denotes. Circumlocution and ambiguity were both rescinded by that invention. It is wonderful that Euler did not hit on it: for the rules belonging to such a symbol, he plainly lays down in *Comm. Petrop.* tom. VIII, p. 163, and afterwards more

must be written $P.\frac{\nu}{dx}$, putting $\nu = LG$, and $dx = CE$. Again, $y' = BL$, and by the translation of L to G, $= BG = BL + \nu$. Hence, dy' must be written ν, and $N'.dy'$, $N'\nu$. again $p' = \frac{LI}{dx}$, and by the translation of L to $G = \frac{LI - LG}{dx} = \frac{LI - \nu}{dx}$. Hence, dp' must be written $\frac{-\nu}{dx}$, and $P'.dp'$, $\frac{-P'\nu}{dx}$. Collecting the several terms, we have the whole variation of $(V + V')\,dx$ equal to $\left(P.\frac{\nu}{dx} + N'\nu - \frac{P'\nu}{dx}\right).dx$, or $\left(N' - \frac{P' - P}{dx}\right)\nu.dx$, or $\left(N - \frac{dP}{dx}\right)\nu.dx$, retaining quantities of the same order: lastly, since the variation must equal nothing, we have for cases comprehended within the expression $dV = Mdx + Ndy + Pdp$, this formula of solution,

$$N - \frac{dP}{dx} = 0.$$

By a process similar, but longer on account of the introduction of the quantity ds, Euler deduces from

$$dV = Mdx + Ndy + Pdp + Lds$$

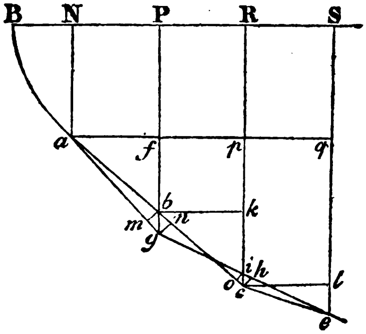
this formula of solution,

$$N - \frac{dP}{dx} + \frac{Ldy}{ds} = 0.$$

---

more fully and distinctly in his tract on Isoperimetricals; and in his last memoir, *Novi Comm. Petrop.* tom. X. p. 54, he says, "Illud problema Isoperimetricum latissimo sensu acceptum, prout id quidem in libro singulari pertractavi; quem qui attente legerit, non dubitabit, quin hujus generis investigationes calculi speciem prorsus singularem postulent, a consuetis analyseos regulis non parum diversam."

The two preceding formulæ have been deduced for st class of problems that involve one property only. 'er's method, although more tedious, is not essen- erent for problems of the second class. These e the variation of three elements of the curve; d accordingly, we must compute the variation in



$(V+V'+V'')\,dx$, when the three elements, instead of ab, bc, ce, become ag, gi, ie; that is, as before we must note the variations in

$$M.dx + N.dy + P.dp + \&c.$$
$$M'.dx + N'.dy' + P'.dp' + \&c.$$
$$M''.dx + N''.dy'' + P''.dp'' + \&c.$$

arising from the translation of the points b and c, to g and i, and from the introduction of two arbitrary quantities or variations such as bg, ci, and thence will result an equation of the form $A.bg - B.ci = 0$, which compared with the equation, $R.bg - (R+dR).ci = 0$, which Euler had previously obtained*, would give $\frac{dR}{R}=\frac{B-A}{A}$, whence R is known.

In the formula $dV=Mdx+Ndy+Pdp$ (forming the equation $A.bg-B.ci$, according to the precepts just given,)

$$A=N'.dx-dP$$
$$B=N''.dx-dP'$$

consequently, $\frac{dR}{R}=\frac{d(N'dx-dP)}{N'dx-dP}$, and $R=N'dx-dP$, or $=dx\left(N-\frac{dP}{dx}\right)$.

This is an equation derived from one property $\int Vdx$: deduce a similar equation from the other $\int Wdx$; for instance, such as $\varrho=dx\left(\nu-\frac{d\pi}{dx}\right)$, and the equation to the curve will be

$$R+a\varrho=0, \text{ or } \left(N-\frac{dP}{dx}\right)+a\left(\nu-\frac{d\pi}{dx}\right)=0;$$
$$\text{or } N+a\nu-\frac{d(P+a\pi)}{dx}=0;$$

and since this form is like that of $N-\frac{dP}{dx}$, instead of two operations for finding the values of R, R', from $\int Vdx=$ a maximum, and $\int Wdx$ = a constant quantity, we may substitute one, and deduce the resulting equation from

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* P. 37.

$$\int V dx + a \int W dx,$$
$$\text{or } \int (V + aW)\, dx,$$

and for this great simplicity introduced into the calculus, we are indebted solely to Euler.

For the sake of stating and illustrating Euler's method, we have taken a simple form to represent dV, that is, $Mdx + Ndy + Pdp$: but Euler introduces a term Lds, which, for problems of the second class, renders the calculation longer, and the result more complicated: in this case, however,

$$\frac{dR}{R} = \frac{-d^2P + L'dx.d\epsilon + dxd(L'\epsilon + N)}{-dP + dx\,(L'\epsilon + N)} \quad \left[\epsilon = \frac{dy}{ds}\right]$$
$$= \frac{L'dx.d\epsilon}{-dP + dx(L'\epsilon + N)} + \frac{d[-dP + dx(L'\epsilon + N)]}{-dP + dx(L'\epsilon + N)};$$

and, $R = e^{\frac{\int Ldx.d\epsilon}{-dP + dx\,(L\epsilon + N)}} \times (-dP + dx\ [L\epsilon + M])$.

This memoir of Euler contains almost all the matter which is to be found in his subsequent tract on Isoperimetricals: but, the matter is ill arranged: considerable perplexity, and some error, is introduced by the term Lds: the cases in which it enters, are mixed with those in which $L = 0$. These latter, in the above mentioned tract, the author separately considers; and even here he notices, that as from $dV = Mdx + Ndy$, $dV = Mdx + Ndy + Pdp$: the resulting formulæ of solution are, $N = 0$, and $N - \frac{dP}{dx} = 0$, so if we put

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + \&c.,$$
$$\left[p = \frac{dy}{dx},\ q = \frac{dp}{dx},\ r = \frac{dq}{dx},\ \&c.\right]$$

then the resulting formula will be

$$N-\frac{dP}{dx}+\frac{d^2Q}{dx^2}-\frac{d^3R}{dx^3}+\&c.=0$$

which, by the principle stated in page 52, will belong to problems of all classes involving definite expressions. This last formula, remarkable for the simplicity of its law, supersedes Euler's table of the values of P*, given in his former memoir. In fact, all problems, in which integral expressions do not enter, are solved by it; and although Euler himself, and afterwards Lagrange, very materially simplified, and expedited its proof, yet, as a formula of solution, it still remains as a final result of all researches on this subject.

Problems are classed according to the number of properties which they involve †; and the number determines that of the ordinates that must be made to vary. But, problems might be distinguished also from each other, by the order of differentials which they involve; and then the order would determine the number of terms to be used in the formula $dV=Mdx+Ndy+Pdp$ + &c. thus, if the curve were required, in which $\int(ax-y^2)y\,dx$ is a maximum, we have $V=(ax-y^2)y$, $dV=ay.dx+(ax-3y^2)\,dy$: consequently, two terms are sufficient, namely, Mdx, Ndy, and the formula of solution is

$$N=0, \text{ or } ax-3y^2=0.$$

This problem is, according to Euler's classification, of the first class, and might be said also to be of the first order.

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* See p. 41.

† See p. 35.

If the *brachystochrone,* with no limitation, were required, since the time

$$=\int\frac{ds}{\text{vel}^y.}=\int\frac{ds}{\sqrt{y}}=\int\frac{\sqrt{(dx^2+dy^2)}}{\sqrt{y}}=\int dx\frac{\sqrt{(1+p^2)}}{\sqrt{y}},$$

by comparing it with $\int Vdx$, we have $V=\frac{\sqrt{(1+p^2)}}{\sqrt{y}}$ *,

and thence $dV=-\frac{\sqrt{(1+p^2)}}{2.y^{\frac{3}{2}}}.dy+\frac{p}{\sqrt{y(1+p^2)}}dp$; so that, the formula $Mdx+Ndy+Pdp$, or three terms are sufficient, one of which $Mdx=0$.

This problem of the first class, might be said to be of the second order.

Again, the solid of least resistance is of the same class and same order with the preceding problem; for since $\int\frac{y.dy^3}{dx^2+dy^2}=$ a minimum $=\int Vdx$, by comparison, we have, $V=\frac{yp^3}{1+p^2}$, and $dV=\frac{p^3}{1+p^2}.dy+\frac{3p^2+p^4}{(1+p^2)^2}.y\,dp$.

But, if a curve were required, in which $\int\frac{d^2y}{ds}$ should be a minimum, then since $Vdx=\frac{d^2y}{ds}=\frac{d^2y}{dx^2}\cdot\frac{dx}{\sqrt{(1+p^2)}}$

$$V=\frac{q}{\sqrt{(1+p^2)}},\text{ and }dV=-\frac{pq}{(1+p^2)^{\frac{3}{2}}}.dp+\frac{1}{\sqrt{(1+p^2)}}.dq$$

so that this problem would require four terms, or this formula

$$dV=Mdx+Ndy+Pdp+Qdq;$$

(in which M and $N=0$), and therefore might be said to be of the third order, and, according to Euler's classification, would be of the first class, if it depended on the minimum property alone.

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* See p. 51.

The class of problems then determines the number of ordinates that must be made to vary; but, on what might be called their order, would depend the number of ordinates to be taken account of, in the computation of the variations. For, as more terms of the series $Mdx+Ndy+$&c. enter, more ordinates are required to estimate the variations of the terms: for instance, let $dV=Mdx+Ndy+Pdp$: then if the problem is of the first class, the variation of one ordinate is sufficient. Let* $BL=y'$ be the ordinate that varies, and let the two adjacent ordinates be y, y'', then it is plain, from the subjoined table, that three ordinates y, y', y'' are sufficient for the computation of the variations in

$$Mdx+Ndy+Pdp, \text{ and in } M'dx+N'dy'+P'.dp',$$

$y = AC$	$p = \frac{dy}{dx} = \frac{EL}{dx}$	Variation in $p = \frac{\nu}{dx}$.
$y' = BL$	$p' = \frac{dy'}{dx} = \frac{LI}{dx}$	variation in $p' = -\frac{\nu}{dx}$.
$y'' = KD$	$p'' = \frac{dy''}{dx}$	

The introduction of other ordinates, such as y''', y^{iv}, &c. or $y_{\prime}\, y_{\prime\prime}$ &c. on either side, is plainly unnecessary.

If, however, $dq=\frac{d^2p}{dx^2}=\frac{d^3y}{dx^3}$ enters into the form, or if

$$dV= Mdx + Ndy + Pdp + Qdq,$$

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* See Fig. p. 52.

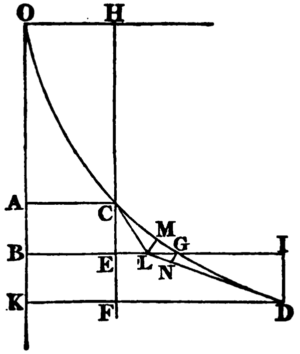
then three ordinates are not sufficient: an additional one $y_{\prime}$ must be taken account of: for the element $V_{\prime}dx$ of the area analogous to Vdx, and immediately preceding it, is $M_{\prime}dx + N_{\prime}dy_{\prime} + P_{\prime}dp_{\prime} + Q_{\prime}.dq_{\prime}$; now, the last term $Q_{\prime}dq_{\prime}$ of this formula, is affected by the increment ι of the ordinate y'; for, since $q = \frac{dp}{dx} = \frac{p'-p}{dx}$, $q_{\prime} = \frac{p-p_{\prime}}{dx}$; $\therefore$ $q_{\prime}$ is varied, if p is varied; and p is varied, if y' is. Hence we must introduce into the computation an additional ordinate $y_{\prime}$; but that is sufficient; for none of the terms representing $V''.dx$, $V'''.dx$, &c. will be affected by the variation of the ordinate y'.

What has been said relates to those cases in which V involves no indefinite expressions; for if a term, such as Lds be introduced, a fourth ordinate must be taken account of. If the problem is of the second class, then two ordinates must be made to vary, and the number of ordinates that must be taken account of is easily determined from the preceding reasonings. We shall have occasion, however, to notice this case, when we speak of Euler's treatise on Isoperimetrical problems.

In the conclusion of his memoir, Euler notices, that his methods fail, or require peculiar artifices, if the quantity V in $\int Vdx$, contains the arc *s*, or other integral quantities. For, in that case, the principle of the whole curve possessing a certain property of maximum or minimum, if a portion of it does, is not true. For if V contains an integral expression, then the curve *OCD* may contain $\int Vdx$ a maximum or minimum, although no separate portion of it has that property. If, indeed,

$$dV = Mdx + Ndy + P\,dp + \&c.$$

then, since the quantities x, y, $\frac{dy}{dx}$, or p depend on the point C, the part CLD may be changed at pleasure, without any change being produced in OC; but if V contains an integral, then if CLD is changed, OC will take another value from $\int V dx$. In these cases then, the preceding methods of Euler, deduced from assuming a portion of the curve possessing the requisite property, fail: so likewise do the methods of Bernoulli and Taylor, which are founded on the same principle *.



There are cases, however, Euler observes, in which

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* Bernoulli's has been already mentioned; and Taylor, in his Methodus Incrementorum, p. 67. prefixes to his solution, a similar one. Its truth depends, as it has been said, on the two portions adjacent to CLD not varying whilst CLD varies, which is to be granted, when $\int V dx$ contains no integral.

the ordinary solutions may be applied; although the quantity $\int V dx$ contains an integral: and these happen when the integral is by the conditions common, or the same to all curves, amongst which a required curve is to be found. Included in these cases, are the Isoperimetrical problems of the Bernoullis; for there s, the integral of $dx\sqrt{\left(1+\frac{dy^2}{dx^2}\right)}$ is given: had it not been given, or if, instead of the Isoperimetrical property, another had been substituted, the solution of the second case proposed in James Bernoulli's *programma,* would have been faulty.

Very important objects were obtained by Euler in this memoir. The solution of problems involving differentials of any order; the invention of a formula including his former formulæ, which, to the number of twenty-four, he had inserted in a table; the partial solution of problems involving integral expressions; the establishment of his theorems and formulæ by easier processes *.

An author is usually, more than justly, fond of his last inventions: and Euler, by this memoir, thought he had nearly perfected the method of solving Isoperimetrical problems*: yet his methods were not without their imperfections. They afforded no general solutions of problems involving integral expressions; and, erroneous solutions, when the differential function depended on a quantity given solely by a differential equation not generally integrable: and the cause of these imperfections was the assumption of the principle, that the whole curve will be endowed with the property of maximum or minimum, if any portion whatever of it possess the same [see p. 52, 62.].

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* "Non solum faciliorem quam ante detexi viam ad solutiones hujusmodi problematum perveniendi, sed etiam omnes 24 formulas, quas ante tractaveram, in unicam sum complexus, &c." *Comm. Petrop.* tom. VIII. p. 159.

We are now led, both according to the order of historical and scientific succession, to the consideration of Euler's tract on Isoperimetrical problems.

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* "Triplici ratione illam priorem methodum ad majorem perfectionis gradum sum evecturus." *Comm. Petrop.* tom. VIII. p. 175.

CHAP. V.

Euler's Tract, entitled "Methodus inveniendi Lineas Curvas Proprietate Maximi Minimive gaudentes"—Distribution of Cases into absolute and relative Maxima and Minima—Rules for finding the Increment of Quantities dependent on their varied State—Formulæ of Solution.

THIS Work appeared in the year 1744, about three years after the publication of his last memoir *, on the same subject. It was intended, and, with a few exceptions, it must be conceded, to be a complete treatise; containing essentially all the requisite methods of solution, with great abundance and variety of examples and illustrations. There is wanting, however, to make it a perfect work, and on the subject, the best extant, a new algorithm; a more compendious process of establishing the theorems; and certain supplemental formulæ, that determine, not the nature of the curve, if a curve be the object of enquiry, but the conditions according to which it must be drawn. These desiderata were afterwards supplied by the fertile genius of Lagrange.

The former memoir contained, as it has been already stated, abundance of valuable matter, but ill arranged. The distribution and arrangement, however, of the present work is extremely luminous and regular. Absolute maxima and minima are first treated of, which concern curves that are to be determined solely by the property of maximum or minimum; such a curve is the *brachystochrone* *, which has the property of the least time, out of all curves whatever that can be drawn between two given points. The curve generating by its rotation round its axis the solid of least resistance, is another.

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* The volume of the Commentaries of Petersburg, in which this memoir is inserted, is said to be for the year 1736; which determines nothing concerning the dates in which the memoirs contained in it were written. Several of them, however, were undoubtedly written after 1736; since they contain accounts of observations made in 1740.

If $\int V dx$ be the analytical expression of the maximum or minimum, V may contain either determinate or indeterminate quantities, such as integrals. Euler first considers the former cases, that is, when V contains only quantities, such as x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c. which are plainly determinate quantities, that is, of assignable value, when x or y is given.

After absolute, relative maxima and minima are treated of: these relate to curves that are to be determined not solely by the maximum property, but conjointly by that and other properties. Such a curve is the *brachystochrone* †, when the property of equal length becomes an additional condition; that is, when the curve of quickest descent is required, not amongst all curves whatever, that can be drawn between two given points, but only amongst those that are of a given length: such also is the *brachystochrone*, when a third condition, that of equal area, is added ‡.

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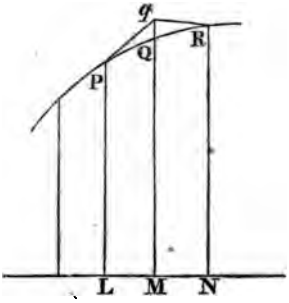
* See pp. 4, 9, 34. † See p. 18. ‡ See p. 47.

In these cases of relative maxima and minima, the quantity V, when $\int V dx$ represents a property, may, or may not, include integral expressions: and since, by an artifice like that which we have stated*, Euler reduces all questions in which are involved two or more properties, analytically expressed by $\int V dx$, $\int Y dx$, $\int X dx$, to this form,

$$\int V dx + a\int Y dx + b\int X dx + \&c.$$

the determination of all cases is reduced, ultimately, to that of an absolute maximum or minimum.

The method employed by Euler in treating the first and simplest cases of absolute maxima and minima, is similar to that used by him in the eighth volume of the Commentaries of Petersburg: thus, suppose the expression of the maximum property to be $\int V dx$, and V to be a determinate function† of x, y, $\frac{dy}{dx}$, &c.; then, as it has



appeared, the property belongs equally to the curve and to its element. Let PQR be a curve, and let its ordinates LP, MQ, NR, &c. be y', y'', y''', &c. and $Qq=\nu$; now $\int V dx$ is to be a minimum or maximum; suppose $\int V_{\backslash}.dx$ to be its value up to the ordinate y, the one immediately preceding LP; then, beyond that ordinate, to the right of y, the value will be increased by $Vdx+V'.dx+V''.dx+$&c.; so that, the whole value of $\int Vdx$ will be

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* P. 56. † P. 62.

$$\int V_{\backslash}.dx + (V+V'+V''+\text{\&c.})\,dx$$

a maximum or minimum, and in which, for the reasons just assigned,

$(V+V'+V''+\text{\&c.})\,dx$ must be a maximum or minimum, that is, $(dV+dV'+dV''+\text{\&c.})\,dx$ must be put equal to nothing*, when the changes arising in dV, dV', &c. from the translation of the point Q to q, or from the increment or *variation* ν are properly expressed; or, if $dV=Mdx+Ndy+Pdp+Qdq$, when the changes or variations of

$$dx\,(Mdx+Ndy+Pdp+Qdq) \qquad [1]$$
$$dx\,(M'dx+N'.dy'+P'.dp'+Q'.dq') \qquad [2]$$
$$dx\,(M''dx+N''.dy''+P''dp''+Q''.dq'') \qquad [3]$$

are expressed.

Now in the Figure p. 67. the ordinate y'' is changed by the quantity ν, and since

$$p=\frac{y'-y}{dx}, \quad q=\frac{p'-p}{dx}=\frac{y''-2y'+y}{dx^2},$$
$$p'=\frac{y''-y'}{dx}, \quad q'=\frac{p''-p'}{dx}=\frac{y'''-2y''+y'}{dx^2}, \qquad [a]$$
$$p''=\frac{y'''-y''}{dx}, \quad q''=\frac{p'''-p''}{dx}=\frac{y^{\text{iv}}-2y'''+y''}{dx^2},$$

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* See p. 53.

it is easy to see the change, or increment, or variation produced in p, p', &c. q, q', &c. from the change ν in y'', which is the sole ordinate that admits of any: thus, the variation* in p is $=0$; in p' is $\frac{+\nu}{dx}$; in p'' is $\frac{-\nu}{dx}$; in q is $\frac{\nu}{dx^2}$; in q' is $\frac{-2\nu}{dx}$; in q'' is $\frac{\nu}{dx^2}$.

Hence, substituting in the three expressions [1], [2], [3],

the variation in [1] is $\frac{Q.\nu}{dx^2}\,dx$;

. . . . in [2] is $\left(P'.\frac{\nu}{dx}-\frac{Q'.2\nu}{dx^2}\right)dx$;

. . . . in [3] is $\left(N''.\nu-P''.\frac{\nu}{dx}+Q''.\frac{\nu}{dx^2}\right)dx$.

Hence, the whole variation is

$$\left(N''-\frac{P''-P'}{dx}+\frac{Q''-2Q'+Q}{dx^2}\right)dx.\nu=0;$$

but, $\frac{P''-P'}{dx}=\frac{dP'}{dx}$ (the increment of P' being diminished in infinitum,)

also $\frac{Q''-2Q'+Q}{dx^2}=$

$$\frac{(Q+2dQ+d^2Q)-2(Q+dQ)+Q}{dx^2}=\frac{d^2Q}{dx^2}.$$

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* Variation, now used technically, is not in this Tract used so by Euler.

Hence, retaining, for the sake of homogeneity, quantities of the same order,

$$N-\frac{dP}{dx}+\frac{d^2Q}{dx^2}=0,$$

which formula will solve all questions of absolute maxima that do not involve integral expressions, or differential expressions, of a higher order than dq, or $\frac{d^2y}{dx^2}$; and by means of the principle and formula stated in p. 56, will solve all questions of relative maxima that neither involve integral, nor differential expressions of a higher order than $\frac{d^2y}{dx^2}$.

There are two points, in the preceding process, that require explanation. The increment or variation ι is attributed not to the ordinate y or y', but to y''; and three values of dV, such as $Mdx+Ndy+Pdp+Qdq$, only were taken. With regard to the first point, the value of $\int Vdx$, that is, $\int V_{\backslash}dx$, from the left hand up to the ordinate y, is supposed to be determined, and not to be affected by any change in that portion of the curve that lies to the right of y; now if y'' be changed, this happens, or $\int V_{\backslash}dx$ is not affected: for, an element of it to the left of y and immediately preceding y, would be, analogously to the mode of expression used,

$$M_{\backslash}dx+N_{\backslash}.dy_{\backslash}+P_{\backslash}.dp_{\backslash}+Q_{\backslash}.dq_{\backslash}$$

and according to the table [a] p. 68, we should have

$$p_{\backslash}=\frac{y-y_{\backslash}}{dx},\ q_{\backslash}=\frac{y'-2y+y_{\backslash}}{dx^2}$$

neither of which quantities depend on y'', the ordinate that is changed; and consequently, these quantities would suffer no change or variation, and therefore $\int V_{\backslash} dx$ will not be affected. But, if to y' instead of y'' the increment ν had been attributed, then $q_{\backslash}$ would have been changed, and its variation would have been $\frac{\nu}{dx^2}$.

This explanation includes, in fact, that of the second point: if a fourth horizontal series

$$M'''.dx + N'''.dy''' + P'''.dp''' + Q'''.dq'''$$

had been taken, then in each term, there would have been no variation from the variation ν of the ordinate y''. For, according to the table [a]

$p''' = \frac{y^{\text{IV}} - y'''}{dx}$, $q''' = \frac{y^{\text{V}} - 2y^{\text{IV}} + y'''}{dx^2}$, which quantities are independent of y''.

If the quantity V involve differentials of a higher order than d^2y, the variation ν must be attributed to an ordinate more remote from y than y''; and more horizontal rows like [1], [2], [3], p. 68, must be taken into the computation. We may easily learn from the preceding explanation, the exact process that must be instituted; for instance, suppose

$$[b] \quad dV = Mdx + Ndy + Pdp + Qdq + Rdr$$

$$\left[r = \frac{dq}{dx} = \frac{d^2p}{dx^2} = \frac{d^3y}{dx^3}\right]$$

then, forming r', r'', &c. as p', p'', q', q'', &c. were formed in table [a] p. 68, we have

$$r=\frac{q'-q}{dx}=\frac{y'''-3y''+3y'-y}{dx^3},$$

$$r'=\frac{q''-q'}{dx}=\frac{y^{\text{iv}}-3y'''+3y''-y'}{dx^3},$$

$$r''=\frac{q'''-q''}{dx}=\frac{y^{\text{v}}-3y^{\text{iv}}+3y'''-y''}{dx^3},$$

$$r'''=\frac{q^{\text{iv}}-q'''}{dx}=\frac{y^{\text{vi}}-3y^{\text{v}}+3y^{\text{iv}}-y'''}{dx^3};$$

now, if ν be attributed to y'', then the preceding value of r, or $r_{\backprime}=\frac{y''-3y'+3y-y_{\backprime}}{dx^3}$ would be affected by the change in r; therefore, we must, instead of y'', make the next ordinate, or y''', to vary by the quantity ν, and then r, and consequently, $\int V_{\backprime}\,dx$ will not be affected by any change in the curve that lies to the right of y: again, four rows like to [*b*] p. 71, are sufficient; for, if we introduce a fifth, since y^{iv}, p^{iv}, q^{iv}, r^{iv} do not involve y'', no change would be introduced in it, or its variation, arising from ν, would be nothing.

If the several variations in the four values of dV be collected and reduced, after the manner given in p. 69, the result will be

$$\left(N-\frac{dP}{dx}+\frac{d^2Q}{dx^2}-\frac{d^3R}{dx^3}\right).dx.\nu$$

which, by the nature of the question, that is, by the property of maximum, must equal nothing.

The above mode of computing that peculiar increment of $\int V dx$, which arises from a change in the curve itself, and which Lagrange technically, and for distinction, called the variation, is sufficiently plain and direct. The rule too that Euler gives *, is only in different terms, the very rule now in use; for instance, to find the variation of

$$y' \surd(1+p^2)$$

take, says he, the differential of this quantity as in the ordinary calculus; and then, instead of the differentials of y', p, substitute those peculiar differentials, that is, variations, that arise from a change in one of the ordinates (y'); accordingly, the common differential being

$$(dy') \times \surd(1+p^2) + \frac{y'p}{\surd(1+p^2)}(dp)$$

the peculiar differential, or variation, if $(dy') = \nu$ and $(dp) = \frac{\nu}{dx}$, will be

$$\nu \surd(1+p^2) + \frac{y'p}{\surd(1+p^2)} \cdot \frac{\nu}{dx}.$$

In the preceding formula, V contains definite quantities, such as x, y, dy, &c. but it may contain an integral expression, or be of this form $T\int Zdx$: in such case, Euler employs a method similar to his preceding one; but, the method becomes very complicated: for then corresponding to V would be $T\int Zdx$, and since the consecutive value of T would be T', and of $\int Zdx$, $\int Zdx + Z.dx$, the quantity corresponding

to V' would be $T'\int Zdx + T'.Zdx$,

to V'' . . . . . . $T''\int Zdx + T''Zdx + T''.Z'dx$,

and then from these expressions, dV, dV', &c. must be expressed, and their values put down, such as arise from the variation ν of any ordinate as $y^{(n)}$.

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* *Methodus Inveniendi*, p. 33, &c.

It will be easily perceived from what is already said, that this process must be very tedious; and it is not necessary more fully to explain it, since the method, with regard to its principle, contains nothing either novel or abstruse, and as a method of computation has been superseded by another more regular and concise.

Since $s = \int \surd(dx^2 + dy^2) = \int dx \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}$ or since s is an integral, if

$$dV = Mdx + Ndy + Lds,$$

the formula for the peculiar differential or variation of $\int Vdx$ would be given by the preceding method; which formula is to be adopted to the exclusion of those other given (dV having the same expression) by Euler in the eighth volume of the Petersburg Commentaries.

V has been supposed to contain an integral $\int Zdx$, and we may go on farther, and suppose Z also to contain an integral such as $\int Xdx$, or V to contain a double integral, and so on: the methods to be pursued for such cases will be similar to the preceding: they will, however, from their complication shew how inconvenient Euler's process is; although, if we attend to its management by the author himself, we cannot help admiring the singular art and dexterity with which, from a perplexed mass of symbols, he at length extricates compact and simple formulæ of solution.

Euler in the sixth volume of the Commentaries of Petersburg reduced problems of relative maxima and minima to a dependance on as many similar equations, as the properties proposed. For instance, if the curve required was to possess two properties, the equations would be of the form $R.bg-S.ci$*, in which $S=R+dR$. But, Euler gives there no satisfactory nor general proof, that $S=R+dR$; but contents himself with saying, that from the analytical process, such will appear to be the value of S; and if S should not be under the proper form, he then directs the computist so to reduce the equation $R.bg-S.ci$, that the coefficient analogous to S, shall equal $R+dR$.

In the present Work, however, he gives a much more satisfactory proof of this important principle: the nature of which will be easily understood from one or two simple examples. Let

$$dV = Mdx + Ndy + Pdp,$$

and let ν be the variation of the ordinate y'' and ω that of y'; ν and ω representing quantities such as bg and ci are in Bernoulli's and Euler's former method†: then since [see p.68.]

$$p = \frac{y'-y}{dx}, \quad p' = \frac{y''-y'}{dx}, \quad p'' = \frac{y'''-y''}{dx}$$

the variation of $p = \frac{\omega}{dx}$, of $p' = \frac{\nu-\omega}{dx}$, of $p'' = -\frac{\nu}{dx}$

consequently, the variation of $dV.dx = \frac{P\omega}{dx}.dx$

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* P. 37. † Pp. 15, 36.

the variation of $dV'.dx = \left(N'\omega + P'.\frac{\nu - \omega}{dx}\right)dx$

. . . . of $dV''.dx = \left(N''.\nu - P''.\frac{\nu}{dx}\right)dx.$

Hence, the whole variation is [see pp. 55, 68.]

$$\left(N' - \frac{P' - P}{dx}\right).dx.\omega + \left(N'' - \frac{P'' - P'}{dx}\right).dx.\nu.$$

Now, N'' is the consecutive value of N', or $= N' + dN'$ $P'' - P'$ is the consecutive value of $P' - P$, or $= (P' - P) + d(P' - P)$; consequently, if we make $N' - \frac{P' - P}{dx} = R$, the coefficient of the second term, or $N'' - \frac{P'' - P'}{dx}$, will be $R + dR$. Hence, the form of the resulting equation, will be

$$R\omega + (R + dR)\,\nu.$$

Again, suppose

$$dV = Mdx + Ndy + Pdp + Qdq;$$

here, if the increments ν and ω are attributed to y'' and y', we must introduce an ordinate $y_{\backslash}$ preceding y; since $q_{\backslash} = \frac{y' - 2y + y_{\backslash}}{dx^2}$. Hence, referring to the values of p, q, given, p. 68, and operating as before;

variation of $p_{\backslash} = 0$, of $p = \frac{\omega}{dx}$, of $p' = \frac{\nu - \omega}{dx}$, of $p'' = \frac{-\nu}{dx}$

of $q_{\backslash} = \frac{\omega}{dx^2}$, of $q = \frac{\nu - 2\omega}{dx^2}$, of $q' = \frac{-2\nu + \omega}{dx^2}$, of $q'' = \frac{\nu}{dx^2}$

Hence the variat^{n}. of $dV_{\backprime}.dx = Q_{\backprime}\frac{\omega}{dx^2}.dx$,

$$\ldots\ldots \text{ of } dV.dx = \left(P\frac{\omega}{dx} + Q.\frac{\nu - 2\omega}{dx^2}\right).dx$$

$$\ldots\ldots \text{ of } dV'.dx = \left(N'\omega + P'.\frac{\nu - \omega}{dx} + Q'.\frac{\omega - 2\nu}{dx^2}\right) dx,$$

$$\ldots\ldots \text{ of } dV''.dx = \left(N''.\nu - P''\frac{\nu}{dx} + Q''.\frac{\nu}{dx^2}\right).dx$$

or, collecting the terms affected with ω, and ν, the variation is

$$\left(N' - \frac{P' - P}{dx} + \frac{Q_{\backprime} - 2Q + Q'}{dx^2}\right)\omega.dx + \left(N'' - \frac{P'' - P'}{dx} + \frac{Q - 2Q' + Q''}{dx^2}\right)\nu.dx;$$

where it is plain, the coefficient of the term involving νdx, is consecutive to the coefficient of the term involving $\omega.dx$; or is $R + dR$, if the coefficient of the first term is R; hence, as before, the resulting formula is

$$R.\omega + (R + dR).\nu.$$

The truth of this result may be also easily perceived from the form which the variations of p, p', p'', and of q, q', q'', &c. necessarily assume from their expressions in terms of the consecutive ordinates, y, y', y'', &c.

We may here again notice the important result, which Euler drew from the above formula. Since $R\omega + (R + dR)\nu$ is derived from a property, of maximum or other, represented by $\int Vdx$; if another property $\int Ydx$ be added, a similar equation, $Q\omega + (Q + dQ)\,\nu$ may be derived: and exterminating ν, the resulting equation is

$$R + aQ = 0; \text{ that is,}$$

$$\text{if } V = M dx + N dy + P dp + \&c.$$
$$\text{and } Y = \mu . dx + \nu . dy + \pi . dp + \&c.$$
$$\text{we shall have } R + aQ = \left(N - \frac{dP}{dx} + \&c.\right) + a\left(\nu - \frac{d\pi}{dx} + \&c.\right)$$
$$= (N + a\nu) - \frac{d(P + a\pi)}{dx} + \&c.;$$

that is, the same formula that would result, if the property proposed had been

$$\int V dx + a \int Y . dx.$$

By this, all questions of *relative* are reduced to those of *absolute* maxima and minima: for, similar reasonings and properties hold, when the curve sought instead of two, has three, four, &c. properties: and if such properties be expressed by

$$\int V dx, \int Y dx, \int W dx, \int U dx, \&c.$$

then we must solve the question as one of absolute maximum and minimum; and inquire, what the curve is that has the property expressed by

$$\int V dx + a \int Y dx + b \int W dx + c \int U dx.$$

Euler, besides the cases already mentioned, solves also those, in which V contains quantities, neither determinate such as x, y, p, &c. nor integrals; but expressed solely under the forms of differential equations. What we have given however, is sufficient to explain and illustrate Euler's method. The results of that method are, for the practical solution of problems, under a most convenient form. On that head there is nothing to desire. Neither is there any want of perspicuity in the principle or in the conduct of his method. It is the length of the operations attendant on his method, the want of mechanism in his calculus, that are objectionable *. These inconveniences Lagrange removed: but, as in such cases it not unfrequently happens, whilst he rendered the process of calculation more expeditious, he deprived its principles of a considerable portion of their plainness and perspicuity.

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* Euler, we learn from the historical account prefixed to vol. X. of the *Novi Comm. Petrop.*, was sensible of the inconveniences of his method : "Interim tamen ipsa methodus, etiamsi totum negotium satis expedite conficiat, tamen ipsi non satis naturalis est visa, propterea quod vis solutionis tota in consideratione elementorum curvæ investigandæ erat posita, ista vero quæstio facile ita adornari possit, ut ex geometria penitus ad solam analysin puram revocetur, &c . . . Tametsi autem auctor de hoc diu multumque esset meditatus, atque amicis hoc desiderium aperuisset tamen gloria primæ inventionis acutissimo Geometræ Taurinensi Lagrange erat reservata, qui sola analysi usus eandem plane solutionem est adeptus, quam auctor ex considerationibus geometricis elicuerat." *Novi Comm. Petrop.* tom. X. p. 12.

CHAP. VI.

Lagrange's Memoir — Use of an appropriate Symbol to denote the *Variation* of a Quantity — Rules for finding the Variation — New Process of deducing Euler's Formulæ—Invention of new Formulæ.

In the preceding account of Euler's memoir, it cannot have escaped notice, that some ambiguity and much circumlocution took place in describing the process of finding that peculiar increment which depends on the quantity ν, by which the ordinate y or y' is increased.

These inconveniences Lagrange obviated by a simple invention; that of a symbol such as δ[*]; which, analogously to the symbol d of the differential calculus, was to be the means of representing either a quantity or an operation. Thus, the quantities ν and ω, by which y and y' are increased, are symbolically denoted by δy, $\delta y'$: and as d in $d\,(ay^n + by'^m)$ signifies, that the operation of taking the differential of $ay^n + by'^m$ is to be made; so, δ in $\delta\,(ay^n + by'^m)$, signifies that the peculiar differential of $ay^n + by'^m$ is to be taken, when instead of dy, dy'; ν and ω, or δy, $\delta y'$, are to be used.

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[*] "Mais avant tout je dois avertir que comme cette methode exige que les mêmes quantités varient de deux manières differentes, pour ne pas confondre ces variations j'ai introduit dans mes calculs une nouvelle characteristique δ." *Miscell. Taurin.* tom. II. p. 174.

With this change in the system of symbols, and with certain changes in the processes of establishing the fundamental formulæ, Lagrange called his method, a new method of determining the maxima and minima of indefinite integrals. But Euler, resuming the subject in the 10th volume of the New Commentaries of Petersburg, p. 54, called it by a name which still adheres to it, *The Calculus of Variations.*

The variation of $ay^n + by'^m$ is $nay^{n-1}.\delta y + mby'^{m-1}.\delta y'$. For if we recur to the problems solved by Bernoulli and Euler *, we shall find the difference of two contiguous states of a quantity at its maximum or minimum put equal to nothing; not indeed the whole difference, but the first term of the difference; for the terms involving $(bn)^2$, $(co)^2$, &c. † are neglected. Now, the difference in that example, is $\frac{1}{\sqrt{Pb}} - \frac{1}{\sqrt{(Pb+bn)}}$; Pb or y being increased by bn: if y had been increased by dy, the first term of $\frac{1}{\sqrt{Pb}} - \frac{1}{\sqrt{(Pb+dy)}}$ would have been the common differential; consequently the peculiar differential, that which is now to be called the *variation,* differs from the common differential, only inasmuch as bn or ν or δy takes the place of dy. The process then for finding the coefficient of the term involving bn or δy, is precisely the same as that for finding the coefficient of the term involving dy; that is, for finding, what technically* is called the differential coefficient. The rules must consequently be the same: in other words, the rules of the differential calculus, become, with the alteration pointed out, those of the calculus of variations.

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* Pp. 7, 33, &c.

† P. 19.

Hence $\delta(y^n) = ny^{n-1}.\delta y$, which is the first term of $[(y+\delta y)^n - y^n]$ expanded.
Again, $\delta(ay^2+by^4)^{\frac{1}{4}} = \frac{1}{4}(ay^2+by^4)^{-\frac{3}{4}}.(2ay+4by^3)\delta y$.

If δx be the variation of x; then since the differential of xy, or $d(xy)$ is $= xdy + ydx$,
the variation of xy, or $\delta(xy)$ is $= x\delta y + y\delta x$.

If we take the example, p. 73,

since $d[y.\surd(1+p^2)] = dy\surd(1+p^2) + \dfrac{yp}{\surd(1+p^2)}dp$

$$\delta[y\surd(1+p^2)] = \delta y\surd(1+p^2) + \frac{yp}{\surd(1+p^2)}.\delta p$$

In p. 68. $p = \dfrac{y'-y}{dx} = \dfrac{dy}{dx}$;

$$\therefore\ \delta p = \frac{\delta y' - \delta y}{dx} \text{ or} = \frac{\delta dy}{dx}, \text{ but } \frac{\delta y' - \delta y}{dx} = \frac{d\delta y}{dx}.$$

Hence, $\delta p = \dfrac{\delta dy}{dx}$ or $= \dfrac{d\delta y}{dx}$.

If $V = ax^3\surd(1+y^2)$ hyp. log. $p \times e^{mq}$
[e = number whose hyp. log. $= 1$,]

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[*] *Princ. Anal. Calc.* p. 74.

then since $dV = 3ax^2\sqrt{(1+y^2)}\,\text{h.l.}p.e^{mq}.dx + \dfrac{ax^3y\,\text{hl}p.e^{mq}}{\sqrt{(1+y^2)}}dy$

$$+ax^3\sqrt{(1+y^2)}\,\frac{1}{p}\,e^{mq}.dp + ax^3\sqrt{(1+y^2)}\,\text{h.l.}p.e^{mq}m.dq$$

therefore putting M, N, P, Q, for the coefficients of the terms involving dx, dy, dp, dq, we have $\delta V = M.\delta x + N.\delta y + P.\delta p + Q.\delta q$, or expressing M, N, P, Q, as partial differential coefficients *,

$$\delta V = \frac{dV}{dx}\delta x + \frac{dV}{dy}\delta y + \frac{dV}{dp}\delta p + \frac{dV}{dq}\delta q\,\dagger.$$

Hence, generally, whatever be the function V,

if $dV = Mdx + Ndy + Pdp + Qdq +$ &c.

then $\delta V = M\delta x + N\delta y + P\delta p + Q\delta q +$ &c.

Since the processes for finding the differential and variation differ only in the symbols dy, δy, which are arbitrary; it is plain, if both operations are to be performed on an analytical expression, that it is matter of indifference, which operation is performed first: or, if the symbols d, δ, meet together denoting operations, we may, at our pleasure, change their order: for instance $d\delta y$ and δdy are alike significant; for $d\delta y$ means the first term of two successive values of δy, or $=\delta\ (y+dy)-\delta y$ $=\delta y+\delta dy-\delta y=\delta dy$: again, if V for instance be a function of y; then

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* *Princ. Anal. Calc.* p. 79.

† This rule is not only in the first solutions of Isoperimetrical problems, virtually acted upon, but expressed. "Il faut bien remarquer que la difference des fonctions de deux lignes comme RO, RT (y, $y+\delta y$) qui se surpassent d'une quantité TO infiniment petite du second genre, se trouve en *differentiant simplement* la fonction de RO, et en multipliant par TO (δy) *ce* qui en vient, ayant omis les differentielles. Par exemple, si RD (Y) fonction de RO (y) etoit seulement la puissance n de la même RO (y), en quoi consiste le cas de mon frère, c'est a dire, que si la courbe BH etoit une parabole du degré n, alors LM (δY) ou $RO^n - RT^n$ $[(y+\delta y) - y^n]$ seroit$=n$. $RO^{n-1} \times TO$ $(ny^{n-1}.\delta y)$." John Bernoulli. *Acad. des Sciences,* 1706, p. 236; also *Opera,* tom. I. p. 424; and Euler, *Methodus Inveniendi,* p. 33, &c. gives the very rule and method for finding the *variation,* which Lagrange invested with appropriate symbols, [see p. 73.]

$$dV=\frac{dV}{dy}.dy, \text{ and } \delta dV=\frac{d^2V}{dy^2}.\delta y.dy$$

$$\delta V=\frac{dV}{dy}.\delta y, \text{ and } d\delta V=\frac{d^2V}{dy^2}.dy.\delta y;$$

$$\therefore\ \delta dV=d\delta V,$$

or, in a particular instance, when $V=y^n$,

$$\begin{aligned}
d\,(y^n) &= 1^{\text{st}} \text{ term of } \left[(y+dy)^n-y^n\right] = ny^{n-1}.dy\\
\delta d(y^n) &= 1^{\text{st}} \text{ term of } ndy\times\left[(y+\delta y)^{n-1}-y^{n-1}\right]=\\
&\qquad n\,(n-1)y^{n-2}.dy.\delta y,\\
\delta(y^n) &= 1^{\text{st}} \text{ term of } \left[(y+\delta y)^n-y^n\right] = ny^{n-1}.\delta y\\
d\delta\,(y^n) &= 1^{\text{st}} \text{ term of } n\delta y\times\left[(y+dy)^{n-1}-y^{n-1}\right]=\\
&\qquad n(n-1)y^{n-2}.\delta y.dy;\\
&\therefore\ \delta d\,(y^n) = d\delta\,(y^n).
\end{aligned}$$

And, by similar processes, $d^2\delta V=\delta d^2V=\delta ddV=d\delta dV$

$$d^3\delta V=\delta d^3V=d\delta d^2V=d^2\delta dV.$$

This rule is, in Lagrange's method, of the greatest importance; it is an essential part of it. Amongst other uses, it enables us when an integral is concerned, to introduce the symbol δ within the symbol $(\int)$ of the integral: thus, since the symbols d and $\int$ indicate reverse operations,

$V = d\int V$; $\therefore$ $\delta V = \delta d\int V =$ [by Rule, p. 84.] $d\delta\int V$. Hence, taking the integrals on each side

$$\int \delta V = \int d\delta\int V = \delta\int V \; . \; . \; . \; [a]$$

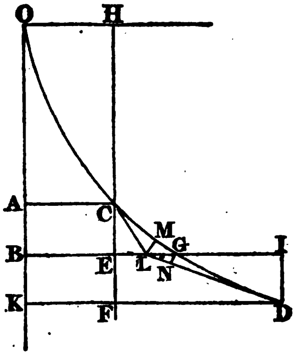
This result may be easily extended to double and treble integrals: for if $V = \int W$, then $\delta V = \delta\int W = \int\delta W$ by [a];

$\therefore$ $\int\delta V = \int\int\delta W$: but $\int\delta V = \delta\int V = \delta\int\int W$, consequently

$$\delta\int\int W = \int\int\delta W.$$

Before we proceed to explain the last improvement made by Lagrange, we will give, under the symbols of his new Algorithm, another solution of the *brachystochrone*, [see pp. 4, 9.]

Let $AC = y$, $BL = y'$, $CL = ds$, $LD = ds'$, $OA = x$, $OB = x'$, $g = 32\frac{1}{2}$ feet; then the velocity $= \sqrt{2gx}$,



hence, $\dfrac{ds}{\sqrt{(2gx)}} + \dfrac{ds'}{\sqrt{(2gx')}} = \text{a minimum}$;

$\therefore$ $\delta\left(\dfrac{ds}{\sqrt{x}} + \dfrac{ds'}{\sqrt{x'}}\right) = 0$, or $\dfrac{d\delta s}{\sqrt{x}} + \dfrac{d\delta s'}{\sqrt{x'}} = 0$,

but $ds = \sqrt{(dx^2 + dy^2)}$; $\therefore$ $d\delta s = \dfrac{dy.d\delta y}{ds}$ and $d\delta s' =$

$\frac{dy'.d\delta y'}{ds'}$; hence, $\frac{dy.d\delta y}{ds\sqrt{x}}+\frac{dy'.d\delta y'}{ds'\sqrt{x'}}=0$ [a]. But

$$KD=AC+dy+dy';$$

$\therefore$ since KD and AC are not changed in the translation of L to G, $dy+dy'$ is constant; $\therefore$ $d\delta y=-d\delta y'$, and substituting in [a], l. 1. $\frac{dy}{ds\sqrt{x}}=\frac{dy'}{ds'\sqrt{x'}}=$ a constant quantity, the property of the cycloid.

This solution is on the same principle, as the first and second, but is less peculiar and geometrical than either; instead of the similar triangles in the second, one of the processes of the new method above described, has been used.

The rules in pp. 84, 85, relate to the second improvement made by Lagrange in the calculus of variations which we shall now describe. If we refer to the method which Euler used for deducing the formula, $N-\frac{dP}{dx}+$&c., we shall perceive that its length and complication arises from the integral $\int Vdx$ being broken down into parcels and elements Vdx, $V'dx$, &c. and from the calculation of the variations in the differential expressions that represent dV, dV', dV'', &c. [pp. 68, 69.]

Lagrange precluded the necessity of this resolution of $\int Vdx$ into its elements, by combining with the preceding *variation* processes, an integral process. This will be understood from the solution of the following problem, in which it is required to find an expression for $\delta\int Vdx$, V being a function of x, y, p, q, r, &c. the law of the formation of p, q, r, &c. being $p=\frac{dy}{dx}$, $q=\frac{dp}{dx}$, $r=\frac{dq}{dx}$, &c.

$\delta\int V dx=\int\delta(V dx)$ [p. 85. l. 3.] $=\int\delta V.dx+\int V d\delta x$.

Now on the principle of this formula, $\int x dy=xy-\int y dx$

$$\int V d\delta x = V.\delta x - \int dV.\delta x.$$

Hence, $\delta\int V dx=V.\delta x+\int(\delta V.dx-dV.\delta x)$

Now since $\delta V=M\delta x+N\delta y+P\delta p+Q\delta q+$&c.

if $dV=Mdx+Ndy+Pdp+Qdq+$&c.

$$\delta V.dx-dV.\delta x=N(\delta y.dx-dy.\delta x)+P(\delta p.dx-dp.\delta x)$$
$$+Q\,(\delta q dx-dq.\delta x)+\text{\&c.}$$
$$=Ndx\,(\delta y-p.\delta x)+Pdx(\delta p-q\delta x)$$
$$+Qdx(\delta q-r\delta x)+\text{\&c.}\left(\text{since } p=\frac{dy}{dx},\ q=\frac{dp}{dx},\ \text{\&c.}\right).$$

Now $p=\frac{dy}{dx}$; $\therefore$ $\delta p=\frac{d\delta y}{dx}-\frac{dy.d\delta x}{dx^2}=\frac{1}{dx}(d\delta y-p.d\delta x)$;

$$\therefore\ \delta p-q\delta x=\frac{1}{dx}(d\delta y-pd\delta x-q\delta x.dx)=$$
$$\frac{1}{dx}(d\delta y-\ pd\delta x-dp.\delta x)=\frac{1}{dx}.d(\delta y-p\delta x)$$

Now since the quantities p, q, r, &c. are formed after the same law, the same relation that subsists between $\delta p-q\delta x$ and $\delta y-p\delta x$, must subsist between $\delta q-r\delta x$ and $\delta p-q\delta x$, and so on; but the relation between the two former quantities is thus expressed:

$$\delta p-q\delta x=\frac{1}{dx}d(\delta y-p\delta x)$$
$$\therefore\ \delta q-r\delta x=\frac{1}{dx}d(\delta p-q\delta x)=*\frac{1}{dx^2}.d^2\,(\delta y-p\delta x)$$

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* dx being constant, otherwise $=\frac{1}{dx}\,d\frac{1}{dx}.d(\delta y-p\delta x)$.

and $\delta r - s\delta x = \frac{1}{dx} d(\delta q - r\delta x) = \frac{1}{dx^3}.d^3(\delta y - p\delta x)$.

Hence, for the purpose of abridgment, putting $\delta y - p\delta x = \delta\omega$, we have,

$$\delta\int V dx = V.\delta x + \int dx\left(N.\delta\omega + \frac{Pd\delta\omega}{dx} + \frac{Qd^2\delta\omega}{dx^2} + \&c.\right).$$

But by Formula, p. 87. l. 3.

$$\int P d\delta\omega = P.\delta\omega - \int dP.\delta\omega,$$

$$\int Q d^2\delta\omega = Qd\delta\omega - \int dQd\delta\omega = Qd\delta\omega - dQ.\delta\omega + \int d^2Q.\delta\omega$$

$$\begin{aligned}\int Rd^3\delta\omega &= Rd^2\delta\omega - \int dRd^2\delta\omega \\ &= Rd^2\delta\omega - dRd\delta\omega + \int d^2R.d\delta\omega, \\ &= R.d^2\delta\omega - dR.d\delta\omega + d^2R\delta\omega - \int d^3R.\delta\omega.\end{aligned}$$

Hence, collecting quantities involving like symbols,

$$\begin{aligned}\delta\int V dx = V\delta x &+ \int dx.\delta\omega\left(N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \frac{d^3R}{dx^3} + \&c.\right) * \\ [A] \qquad &+ \left(P - \frac{dQ}{dx} + \frac{d^2R}{dx^2} - \&c.\right).\delta\omega \\ &+ \left(Q - \frac{dR}{dx} + \&c.\right) d\delta\omega \\ &+ \&c. \qquad + K.\end{aligned}$$

The quantity K represents the sum of the corrections introduced by the integrations.

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* $\frac{dP}{dx}$, $\frac{d^2Q}{dx^2}$, &c. are not partial differential coefficients, but the first, second, &c. entire differentials of P, Q, &c. divided by dx, dx^2, &c. respectively. They ought in strictness, to be written $\frac{1}{dx}.dP$, $\frac{1}{dx^2}.d^2Q$, &c.

If x be supposed to have no *variation*, or if $\delta x = 0$, then $\delta\omega = \delta y - p\delta x = \delta y$; and consequently we shall have the variation of $\delta\int V dx$ by omitting in the preceding form $V\delta x$, and writing δy, $d\delta y$, &c. instead of $\delta\omega$, $d\delta\omega$, &c.

The preceding value of the variation may be supposed to be taken between two limits corresponding to values a, and b, of x. Let y_0, P_0, Q_0, &c. be the values of y, P, Q, &c. at the first limit when $x = a$; and and y_1, P_1, Q_1, &c. when $x = b$;

$$\begin{aligned} \text{then, } \delta\int V dx = {} & V_1 . \delta x_1 - V_0 . \delta x_0 \\ & + \int dx\,\delta\omega \left(N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \&\text{c.}\right) \\ & + \left(P_1 - \frac{dQ_1}{dx} + \&\text{c.}\right)\delta\omega_1, \\ [Aa] \ldots\ldots & - \left(P_0 - \frac{dQ_0}{dx} + \&\text{c.}\right)\delta\omega_0, \\ & + \&\text{c.} \end{aligned}$$

If at the two limits $x = a$, $x = b$, the values of y_0, y_1 are given, then $\delta\omega_1$, $\delta\omega_0$ are both equal to nothing, (since $\delta\omega = \delta y - p\delta x$); and in this case the variation of $\int V dx$ is reduced to the quantity under the integral sign ($\int$), which, in the case of $\int V dx =$ a maximum, since $\delta\int V dx = 0$, must equal nothing; or

$$N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \frac{d^3R}{dx^3} + \&\text{c.} = 0.$$

This last formula, is that which Euler, by the method described in the last Chapter, arrived at*; and it is here deduced on his hypothesis of the evanescence of δy_0, δy_1, &c.

The formula $N-\frac{dP}{dx}+\frac{d^2Q}{dx^2}-$ &c. is not, however, equal 0, solely in the case when δx_0, δx_1, δy_0, δy_1, are equal 0, but also, when these are, from certain equations, assignable quantities. For, the formula [*Aa*] p. 89, is composed of two parts: one, affected by the integral sign, expresses the sum of all the separate variations throughout the whole extent of the curve or integrated quantity; the other part, independant of the integral sign, is affected only by the variations at the extreme points, and therefore cannot by any combination with the other, (which by changing δx and δy may be varied at will) form a sum equal to nothing. Hence, since $\delta \int V dx$ must $=0$; each part separately, the one under the integral sign $\int$, the other not affected by it, must $=0$.

M. Lagrange is the inventor of that part of the general formula which is not affected by the integral sign. Its use, as it will be shewn hereafter, is great and extensive: without it, the solution of problems would be incomplete; for, Euler's formula, that under the integral sign, merely determines in general terms the relation of x and y; undoubtedly the chief, but not the sole object of search.

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* *Methodus Inveniendi*, &c. Prob. V. Lagrange, *Miscell. Taurin.* tom. II., says, " Mais les formules de cet auteur (Euler) sont moins generales que les notres 1° &c.—2°, parcequ'il suppose que le premier et le dernier point de la courbe sont fixés, &c." Euler also, in his memoir subsequent to Lagrange's [*Novi. Comm.* tom X. p. 110.] acknowledges that his first formula did not contain the absolute or definite parts involving $\delta\omega$, $d\delta\omega$, &c. " Neque tamen hæ partes absolutæ frustra sunt inventæ, sed singularem præbent usum, ad quem methodus mea prior, quæ tantum æquationem $N-\frac{dP}{dx}+\&c.=0$ suppeditavit, minus est accommodata; quam ob causam hæc methodus illi longe est anteferenda."

We will now proceed to that, which, in fact, is Lagrange's second problem*.

It is required to find an expression for $\int V dx$, in which $dV = T dt$, dt not being similar to dy, dp, dq, &c., but determined by this equation $t = \int Z dx$.

By p. 87, $\delta \int V dx = V \delta x + \int (\delta V . dx - dV . \delta x) \ldots . [1]$

but $\delta V = T . \delta t$; $\therefore \delta V dx - dV . \delta x = T dx . \delta t - T \delta x . dt$.

again, by [1] $\delta t = \delta \int Z dx = Z . \delta x + \int (\delta Z . dx - dZ . \delta x)$;

$\therefore T dx . \delta t - T \delta x . dt = TZ . dx . \delta x - TZ . dx . \delta x +$

$T dx \int (\delta Z dx - dZ \delta x)$,

$= T dx \int (\delta Z dx - dZ . \delta x)$

and $\therefore \delta \int V dx = V \delta x + \int T dx \int (\delta Z dx - dZ \delta x)$.

Now by virtue of this formula $\int (v du) = vu - \int (u dv)$, if h be put $= \int T dx$, the latter part of the value of $\delta \int V dx$

is, $h \int (\delta Z dx - dZ \delta x) - \int h (\delta Z dx - dZ \delta x)$

Also by Form [A], p. 88. if $dZ = M dx + N dy + P dp + \&c.$

$h \int (\delta Z dx - dZ \delta x) = h \int dx . \delta \omega \left(N - \frac{dP}{dx} + \frac{d^2 Q}{dx^2} - \&c. \right) +$

$h . L \ldots . [2]$

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* *Miscellanea Taurinensia*, tom. II, p. 183.

and $\int h(\delta Z\,dx - dZ\,\delta x) =$

$$\int dx.\delta\omega\left(N - \frac{d(hP)}{dx} + \frac{d^2(hQ)}{dx^2} - \&c.\right) + hL' \dots [3]$$

(L, L' being put for the definite parts,)

Subtracting this from [2] and adding $V\delta x$, we shall have the value of $\delta\int V dx$.

In general, it is required to find the whole variation of $\int V dx$ from $x = 0$, for instance, to $x = a$. In such case, let the whole integral of $T dx = H$; then the part [2], of the

$$\text{variation} = H\int dx\,\delta\omega\left(N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \&c.\right) + HL$$

$$= \int dx\,\delta\omega\left(HN - \frac{d(HP)}{dx} + \frac{d^2(HQ)}{dx^2} - \&c.\right) + HL$$

For since H is constant, $H\int N = \int HN$, and $H.dP = d(HP)$; subtracting therefore, as before, [3] from [2], we have

$$\delta\int V dx = V\delta x$$

$$+ \int dx.\delta\omega\left((H-h)N - \frac{d(H-h)P}{dx} + \frac{d^2(H-h)Q}{dx^2} - \&c.\right)$$

$$[B] \dots\dots + \left((H-h)P - \frac{d(H-h)Q}{dx} + \&c.\right)\delta\omega$$

$$+ \&c.$$

If we put $H - h$, that is, $H - \int T dx = k$, and suppose δx, $\delta\omega$, the variations at the limits, to equal nothing, we shall have

$$\delta\int V dx = \int dx.\delta\omega\left(kN - \frac{d(kP)}{dx} + \frac{d^2(kQ)}{dx^2} - \&c.\right)$$

which is Euler's formula, *Method. Inveniendi,* &c. p. 91, and the subject of his third Chapter.

If $dV = \mu.dx + \nu.dy + \pi.dp + \&c. + Tdt$,
which is Lagrange's case. *Misc. Taur.* tom. II. p. 183.

$$\delta\int Vdx = V\delta x + \int dx.\delta\omega\left(\nu - \frac{d\pi}{dx} + \&c.\right) \ldots [C]$$
$$+ \int dx\delta\omega\left(kN - \frac{d(kP)}{dx} + \&c.\right)$$
$$+ S,$$

S being put for the sum of the definite terms.

If $dt = ds = \sqrt{(dx^2 + dy^2)}$ and $dV = \mu dx + \nu dy + Tds$

$$\delta\int Vdx = \int dx\delta\omega\left[\nu - \frac{d\left([H - \int Tdx]\frac{p}{\sqrt{(1+p^2)}}\right)}{dx}\right] .. [Cc]$$

not taking account of the definite parts;
For, $dt = ds = dx\sqrt{(1+p^2)} = Zdx$; $\therefore Z = \sqrt{(1+p^2)}$ and $dZ = \frac{p}{\sqrt{(1+p^2)}}.dp$; which, compared with the value of dZ [p. 91,] gives $N = 0$, $P = \frac{p}{\sqrt{(1+p^2)}}$, $Q = 0$, &c.

Hence, in the case of maximum or minimum, since $\delta\int Vdx = 0$, there results this equation of solution,

$$\nu.dx - d\left([H - \int Tdx]\frac{p}{\sqrt{(1+p^2)}}\right) = 0;$$

whereas Euler, *Novi Comm.* tom. VI. p. 141, gives this

$$\nu + T\frac{p}{\sqrt{(1+p^2)}} = 0,$$

, answering to M, T to L, and $\frac{p}{\sqrt{(1+p^2)}}$ being $= q$ $= \frac{dy}{ds}$.

This is the case before alluded to [p. 61.]; when it was remarked that Euler's formulæ contained in the sixth and eighth volume of the Petersburg Commentaries, were erroneous, when V contained an integral; which integral in the above instance is s.

In the third problem of Lagrange [p. 185.], it is required to find the variation of $\int V$ when V is given simply by a differential equation involving no differentials of V higher than the first.

Let $dV - X.dx + Udx = 0$ be the differential equation, X being a function of x, y, p, q, &c. and U a function of x, y, p, q, &c. and of V; then taking the differential, and supposing dx to be constant,

$$\begin{aligned} d^2V - \left(\frac{dX}{dx}dx + \frac{dX}{dy}dy + \frac{dX}{dp}dp + \text{\&c.}\right) dx \\ + \left(\frac{dU}{dx}.dx + \frac{dU}{dy}dy + \frac{dU}{dp}dp + \text{\&c.}\right) dx \\ + \frac{dU}{dV}.dV.dx = 0, \end{aligned}$$

$$\text{or } d^2V - d\phi.dx + \frac{dU}{dV}.dV.dx = 0, \quad \ldots\ldots \quad [a]$$

Substituting $d\phi$ for the collection of terms involving dx, dy, dp, &c. Hence, if instead of deducing the differential in the last operation, we suppose the variation to be deduced,

$$d\delta V - \delta\phi.dx + T.\delta V.dx = 0 \qquad \left[T = \frac{dU}{dV}\right]$$

multiply by λ, and $\lambda\, d\delta V - (\lambda\delta\phi)\, dx + (\lambda\, T.\delta V).dx = 0$,
or, $d(\lambda\delta V) - d\lambda.\delta V - (\lambda\delta\phi)\, dx + (\lambda T.\delta V).dx = 0 \ldots [b]$

Assume the sum of the second and fourth terms to $= 0$;
$\therefore\ d\lambda - \lambda T.dx = 0$; and $\frac{d\lambda}{\lambda} = Tdx$, and integrating $\lambda = e^{\int Tdx}$

[e = number whose hyp. log. = 1.]

Hence substituting this value of λ in the sum of the first and third terms,

$$e^{\int Tdx}.\delta V = \int e^{\int Tdx}.\delta\phi.dx;$$

$$\therefore \int\delta V \text{ or } \delta\int V = \int e^{-\int Tdx}.\int e^{\int Tdx}.\delta\phi.dx.$$

Euler, by a different process, solves this problem in the third Chapter of his treatise on Isoperimetricals.

Euler, however, stopped at this problem; and did not, as Lagrange has done, proceed on to those which besides dV, involve d^2V, d^3V, &c. Suppose U to be a function of x, y, p, &c., V and of $\frac{dV}{dx}$; then in taking the differential [a] p. 94. of the equation $dV - Xdx + Udx = 0$, an additional term $\frac{dU}{dv}.d^2V\left(v = \frac{dV}{dx}\right)$ will be introduced, and consequently in the variation, this term $\frac{dU}{dv}.d\delta V$; therefore when the equation is multiplied by λ (for the same process must be used,) there will be an additional term $\lambda T'.d\delta V\left(T' = \frac{dU}{dv}\right) = d(\lambda T'.\delta V) - d(\lambda T') \times \delta V$;
$\therefore$ equation [b] l. 2, would be

$d(\lambda\delta V) - d\lambda.\delta V - \lambda\delta\phi.dx + (\lambda T.\delta V)\,dx + d\,(\lambda T'.\delta V) - d\,(\lambda T')\,\delta V.$

Hence, making the sum of the 2d, 4th, and 6th terms = 0,

$$-\,d\lambda + \lambda.Tdx - d(\lambda T') = 0 \ldots [c]$$

and $d\,[(\lambda + \lambda T')\delta V] - \lambda\delta\phi\,dx = 0 \ldots . \; [d]$

in which two equations, λ must be deduced from the first and substituted in the second.

The process is the same if U besides a function of the former quantities, is also a function of d^2V; for then in the equation of variation there would be introduced a term $T''.d^2\delta V$ $\left(T'' = \frac{dV}{d\omega},\ \omega = \frac{dv}{dx}\right)$; and consequently in the equation multiplied by λ, a term $= \lambda T''.d^2\delta V = d\,(\lambda T''d\delta V) - d(\lambda T'').d\delta V = d\,(\lambda T''.d\delta V) - d^2(\lambda T''.\delta V) + d^2\lambda T'' \times \delta V.$

Hence the equation [c] l. 4. will become

$$-\,d\lambda + \lambda.Tdx - d\,(\lambda T') + d^2.(\lambda T'') = 0,$$

and the equation [d] l. 5.

$d\,[(\lambda + \lambda T' - d(\lambda T''))\,\delta V] - d(\lambda T''.d\delta V) - \lambda\delta\phi\,dx = 0$: and similarly, if differentials of a still higher order are introduced.

In the preceding cases, the maxima and minima depend on a function of one variable quantity alone: but a variable quantity z may be introduced, a function of two others x and y. For instance, suppose it were required to find amongst equal solids, that which is bounded by the least surface, (which is the problem given by

Lagrange in his first Appendix, p. 188.): then, if the points of the surface be referred to three rectangular co-ordinates $x, y, z,$ we have

$$dz = \left(\frac{dz}{dx}\right) dx + \left(\frac{dz}{dy}\right).dy$$

$\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dy}\right)$, being the partial differential coefficients *, we have also the solidity $= \int\int z.dx.dy$, and the surface $=$

$$\int\int dx.dy \sqrt{\left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right]}$$

the integrals being taken, first relatively to x, then relatively to y. The problem therefore analytically expressed is

$$[a] \ldots\ldots \int\int \delta(z.dx.dy)$$

$$+ a\int\int \delta.(dx.dy)\sqrt{\left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right]} = 0,$$

$$\text{now } \delta \sqrt{\left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right]} =$$

$$\left[\frac{dz}{dx}\delta\left(\frac{dz}{dx}\right) + \frac{dz}{dy}\delta\left(\frac{dz}{dy}\right)\right] \cdot \left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right]^{-\frac{1}{2}}$$

$$\text{make } \frac{dz}{dx} \cdot \left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right]^{-\frac{1}{2}} = P,$$

$$\frac{dz}{dy} \left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right]^{-\frac{1}{2}} = Q,$$

$$\text{then } \int P\delta\left(\frac{dz}{dx}\right) = \int P\frac{d\delta z}{dx} = P.\frac{\delta z}{dx} - \int\frac{dP}{dx}\delta z;$$

$$\text{similarly } \int Q\delta\left(\frac{dz}{dy}\right) \;\ldots\ldots\; = Q\,\frac{\delta z}{dy} - \int\frac{dQ}{dy}\delta z.$$

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* *Princ. Anal. Calc.* p. 79.

Hence the equation [a] p. 97. l. 10. becomes

$$\int\int dx.dy.\delta z + a\int P dy.\delta z + a\int Q dx.\delta z$$

$$- a\int\int dx.dy\,\frac{dP}{dx}\delta z - a\int\int dx.dy\,\frac{dQ}{dy}\delta z = 0.$$

Hence, making the sum of the second and third terms, which are affected with one sign of integration, and belong to the extreme points of the curve surface, $= 0$ $Pdy + Qdx = 0$, and consequently the remaining terms $= 0$, or, $1 - a.\frac{dP}{dx} - a.\frac{dQ}{dy} = 0$, [$p$]

now the condition under which

$$Mdx + Ndy$$

is a complete differential, is this *

$$\frac{dM}{dy} = \frac{dN}{dx}$$

Hence, putting $N = x - aP$, $M = aQ$, we have

$$a\frac{dQ}{dy} = 1 - a.\frac{dP}{dx}, \text{ or, } 1 - a\frac{dP}{dx} - a\frac{dQ}{dy} = 0,$$

which is the equation [p] l. 8.; hence, $(x - aP)\,dy + aQdx$ is a complete differen^{l}.: but that, $dz = \frac{dz}{dx}dx + \frac{dz}{dy}dy$ may be real, $\frac{dz}{dx}dx + \frac{dz}{dy}dy$ must be also a complete differential; that is, $\frac{d^2z}{dx.dy}$ must $= \frac{d^2z}{dy.dx}$. These two

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* Woodhouse, *Princ. Anal. Calc.* p. 88 : also Euler, *Calc. Int.* p. 315.

conditions therefore must be fulfilled, in order that the problem may be solved.

If r be the radius of a sphere, and α, β, γ, the three co-ordinates of its center,

$$r=\sqrt{[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]}\,;$$

$$\therefore\ dz=\frac{1}{(z-\gamma)}.[(x-\alpha)\,dx+(y-\beta)\,dy]\,;$$

$$\therefore\ \frac{dz}{dx}=\frac{1}{(z-\gamma)}\,(x-\alpha),\text{ and }\frac{dz}{dy}=\frac{1}{(z-\gamma)}(y-\beta)\,;$$

$\therefore$ (putting for $z-\gamma$ its value) $\frac{d^2z}{dx.dy}=\frac{d^2z}{dy.dx}$ one condition therefore is satisfied:

again $P=\frac{x-\alpha}{r}$, $Q=\frac{y-\beta}{r}$; $\therefore (x-aP)\,dy+aQdx=$ $\left(1-\frac{a}{r}\right)xdy+\frac{a}{r}ydx+\frac{a\alpha}{r}dy-\frac{a\beta}{r}.dx$: which is a complete differential, making $1-\frac{a}{r}=\frac{a}{r}$; $\therefore$ the portion of the sphere corresponding to x, y, z, satisfies the problem.

The formulæ of solution, on which the nature of the curve, or the relation of x and y, depends, were all, excepting the two last, invented by Euler. To Lagrange belongs the merit of having deduced them by neater processes. The latter author, however, is the sole inventor of those definite and absolute formulæ [see p. 88. l. 13, 14.] which are requisite for the complete solution of Isoperimetrical problems.

In his first memoir, Lagrange seems not exactly to have comprehended the nature of these definite formulæ.

He drew some conclusions, which the Chevalier Borda [*] proved to be not general. These defects, however, M. Lagrange remedied in a subsequent memoir in the fourth volume of the Miscellanea Taurinensia: and without acknowledging the detection of the defects, or rather, with a faint endeavour of denying it, extended his formulæ, more accurately applied them, but confirmed the truth of M. Borda's results.

In the same memoir Lagrange considers the subject under a new point of view, and gives a method of solution including all his former ones.

The substance of Lagrange's first researches on this subject have been given, with some deviation, not essential, from their mode. Euler's last manner [†] of treating the subject has been followed. That is commended and adopted by Lagrange in the latest of his publications [‡]. In the same Tract he has resumed the consideration of the general problem, and deduced formulæ applicable to all cases.

This last method of Lagrange's is distinguished rather by its mode of treating the question, than by any thing novel in its principles; and therefore, it will be separately considered in the following Chapter, which the Student who hastens towards the end of the inquiry, may, without inconvenience, pass over.

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[*] *Acad. des Sciences,* 1767. [†] *Novi Comm. Petrop.* tom. X. p. 51.

[‡] *Leçons sur le Calcul des Fonctions.*

CHAP. VII.

Lagrange's general Method of treating Isoperimetrical Problems—Equation of Limits—Cases of relative Maxima and Minima reduced to those of Absolute.

LET V be a function of x, y, p, q, z, p', q', &c.

$$p = \frac{dy}{dx},\quad q = \frac{dp}{dx},\quad p' = \frac{dz}{dx},\quad q' = \frac{dp'}{dx},$$

and let

$$\begin{aligned} dV = {} & Mdx + Ndy + Pdp + Qdq + \&c. \\ & + \nu . dz + \pi . dp' + \sigma . dq' + \&c. \end{aligned}$$

then, if

$$\begin{aligned} Y &= N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \&c. \\ Y' &= P - \frac{dQ}{dx} + \&c. \\ Y'' &= Q - \&c. \\ \&c. &= \&c. \\ Z &= \nu - \frac{d\pi}{dx} + \frac{d^2\sigma}{dx^2} - \&c. \\ Z' &= \pi - \frac{d\sigma}{dx} + \&c. \\ Z'' &= \sigma - \&c. \\ \&c. &= \&c. \end{aligned}$$

we shall have, by processes already described, [pp. 87, 88.]

$$\begin{aligned} \delta \int V dx = {} & V\delta x + \int dx . \delta\omega . Y + \int dx . \delta\omega' Z \\ & + Y' . \delta\omega + Z' . \delta\omega' \\ & + Y'' . d\delta\omega + Z'' . d\delta\omega' \\ & + \&c. \end{aligned}$$

Hence, in the case of a maximum, when $\delta\int V dx = 0$,

$$Y.\delta\omega + Z.\delta\omega' = 0,$$

or $Y\delta y + Z\delta z = 0$, if $\delta x = 0$, since [p. 88.]

$$\delta\omega = \delta y - p\delta x, \text{ and } \delta\omega' = \delta z - p'.\delta x$$

If y, z, are quantities independent of each other, then, $Y = 0$, and $Z = 0$.

But, if the quantities y and z are connected together by an equation, such as $f(x, y, z) = 0$, or $W = 0$, then, supposing x invariable, we have

$$\frac{dW}{dy}.\delta y + \frac{dW}{dz}.\delta z = 0,$$

which combined with the former equation [l. 3.] gives

$$Y.\frac{dW}{dz} - Z.\frac{dW}{dy} = 0.$$

The same result will be obtained, if x be supposed to vary; for then

$$\frac{dW}{dx}\delta x + \frac{dW}{dy}\delta y + \frac{dW}{dz}\delta z = 0,$$

which combined with the common differential equation

$$\frac{dW}{dx}.dx + \frac{dW}{dy}.dy + \frac{dW}{dz}.dz = 0,$$

gives, $\frac{dW}{dy}.\left(\delta y - \frac{dy}{dx}\delta x\right) + \frac{dW}{dz}\left(\delta z - \frac{dz}{dx}\delta x\right) = 0$;

or, $\frac{dW}{dy}.\delta\omega + \frac{dW}{dz}.\delta\omega' = 0$,

which combined with $Y\delta\omega + Z\delta\omega'$, gives

$$Y.\frac{dW}{dz} - Z.\frac{dW}{dy} = 0, \text{ as before.}$$

Hence, this conclusion follows; that the variation of x in no wise affects the general equation of maximum or minimum, but solely the equation at the limits.

The equation at the limits depends on the parts of the formula for $\delta\int V dx$, which are definite and freed from the integral sign, and is

$$\begin{aligned} &= V_1.\delta x_1 - V_0.\delta x_0 \\ &\qquad + Y'_1.\delta\omega_1 + Z'_1.\delta\omega'_1 + \&c. \\ &\qquad - Y'_0.\delta\omega_0 - Z'_0.\delta\omega'_0 + \&c. \end{aligned}$$

V_1, V_0, δx_1, δx_0, &c. representing the values of V, δx, &c. at the end and beginning of the integral.

Instead of eliminating $\delta\omega$, $\delta\omega'$, by the combination of the two equations [p. 102. l. 19, 20.] we may multiply the latter by an indeterminate quantity λ, add it to the first equation, and then determine the resulting equation, by the elimination of λ; thus

$$Y.\delta\omega + Z.\delta\omega' + \lambda\frac{dW}{dy}.\delta\omega + \lambda\frac{dW}{dz}\delta\omega' = 0.$$

Eliminate λ from the two equations

$$Y + \lambda\frac{dW}{dy} = 0, \text{ and } Z + \lambda\frac{dW}{dz} = 0,$$

and there results, as before,

$$Y.\frac{dW}{dz} - Z.\frac{dW}{dy} = 0,$$

which combined with $W = 0$, will give the values of y and z in terms of x.

This last method extended, will afford a general formula of solution; for, suppose W to be a function, not of x, y, z, only, but also of the differentials of these quantities; that is, suppose

$$
\begin{aligned}
dW = {} & M'.dx + N'.dy + P'.dp + Q'.dq + \&c. \\
& + \nu'.dz + \pi'.dp' + \sigma'.dq' + \&c.
\end{aligned}
$$

and let $(Y) = \lambda N' - \dfrac{d(\lambda P')}{dx} + \dfrac{d^2(\lambda Q')}{dx^2} - \&c.$

$$(Y') = \lambda P' - \frac{d(\lambda Q')}{dx} + \&c.$$

&c.

$$(Z) = \lambda \nu' - \frac{d(\lambda \pi')}{dx} + \&c.$$

$$(Z') = \lambda \pi' - \frac{d(\lambda \sigma')}{dx} + \&c.$$

&c.

then we shall have

$$
\begin{aligned}
\delta\!\int\! Vdx + \delta\!\int\!(\lambda W).dx = {} & (V + \lambda W).\delta x \\
& + \int[Y + (Y).dx.\delta\omega] + \int[Z + (Z)dx\,\delta\omega'] \\
& + [Y' + (Y')]\,\delta\omega + [Z' + (Z')]\,\delta\omega' \\
& + \&c.
\end{aligned}
$$

In this formula since $W = 0$, $\lambda W.\delta x = 0$, and since $\delta\omega$, $\delta\omega'$ are independent quantities, we have, on principles such as have been already stated,

$$Y + (Y) = 0, \qquad Z + (Z) = 0,$$

whence λ is to be eliminated; and then by the aid of the equation of condition $W = 0$, y and z are to be determined in terms of x.

This method comprehends all the former ones which were separately instituted [pp. 88, 92.] to determine the variation of $\int V dx$.

1st. When dV contains merely x, y, p, q, &c. and no relation is assigned between x, y, z, then the equation $W = 0$ is not introduced, and the formula becomes $Y = 0$, $Z = 0$;

$$\text{or } N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \&c. = 0,$$

$$\text{and } \nu - \frac{d\pi}{dx} + \frac{d^2\sigma}{dx^2} - \&c. = 0.$$

2dly, If V contains an integral expression; that is, if

$$dV = Mdx + Ndy + Pdp + \&c. + Tdt,$$

in which $t = \int Sdx$, then the equation of condition $W = 0$ becomes $dt - S.dx = 0$, and consequently $\int \left(\frac{dt}{dx} - S\right)dx = a$. Hence, substituting t for z, we have

$$Y = N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \&c.$$

$$Z = T \qquad [\text{since } \nu = T,\ \sigma = 0,\ \&c.]$$

$$-(Y) = \lambda N' - \frac{d(\lambda P')}{dx} + \frac{d^2(\lambda Q')}{dx^2} - \&c.$$

$$[\text{if } dS = M'.dx + N'.dy + \&c.]$$

$$(Z) = -\frac{d\lambda}{dx} \qquad [\text{since } \nu' = 0,\ \pi' = 1,\ \sigma' = 0,\ \&c.]$$

hence we have from the equation $Z + (Z) = 0$.

$$T - \frac{d\lambda}{dx} = 0, \text{ whence, } \lambda = \int Tdx = \int Tdx - H,$$

if $\int Tdx = H$, when $\lambda = 0$: substituting this value of λ in the equation $Y + (Y) = 0$, there results

$N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \&c. + (H - \int T dx)N' - \frac{d[(H - \int T dx)P']}{dx}$
$+ \&c.$ as before, p. 92.

3dly, If $dV - \phi . dx + U . dx = 0$, and U is a function of V, this answers to the equation $W = 0$, and $W = \frac{dV}{dx} - \phi + U$, and $dW = T . dV + d . \frac{dV}{dx} - d\phi$; instead of z, consider V to be the variable quantity, then in

$$dW = d\phi + \nu' . dz + \pi' . dp' + \&c. \quad \nu' = T, \ \pi' = 1,$$

the other coefficients being 0;

$$\text{Hence, } Y = 0, \ (Y) = \lambda N' - \frac{d(\lambda P')}{dx} + \&c.$$

$$Z = 0, \ (Z) = \lambda T - \frac{d\lambda}{dx};$$

$\therefore Z + (Z) = 0$, or $\lambda T - \frac{d\lambda}{dx} = 0$, whence $\lambda = e^{\int T dx}$ as before, p. 95.

The same method applies to equations in which higher differentials than the first are involved; for instance, to equations such as

$$t'' \frac{d^3V}{dx^3} + t' . \frac{d^2V}{dx^2} + t . \frac{dV}{dx} + \phi = 0.$$

In the variation of $\int V dx$, the parts under the integral sign, that is, $Y + (Y)$, and $Z + (Z)$ equal nothing; there remains therefore, supposing $\delta x = 0$,

$$\begin{aligned} \delta \int V dx = & \ [Y' + (Y')]\delta y + [Z' + (Z')]\delta z \\ & + [Y'' + (Y'')]d\delta y + [Z'' + (Z'')]d\delta z \\ & + \&c. \end{aligned}$$

which expression must be substituted in the equation of limits, that is, in

$$\delta U_1 - \delta U_0 = 0,$$

putting $U = \int V dx$.

At the limits, there may be particular relations between x and y; for instance, it may be required to draw the brachystochrone between certain curves. Let $L = 0$ be the equation to the curve at the first point, and $M = 0$ at the last point, that is, let L be a function of x_0, y_0, and M a function of x_1, y_1; then

$$\frac{dL}{dx} . \delta x_0 + \frac{dL}{dy} . \delta y_0 = 0, \text{ and } \frac{dM}{dx} \delta x_1 + \frac{dM}{dy} \delta y_1 = 0;$$

and since, by taking the differential equations, we have

$$\frac{dL}{dx} . dx_0 + \frac{dL}{dy} . dy_0 = 0, \text{ and } \frac{dM}{dx} dx_1 + \frac{dM}{dy} dy_1 = 0,$$

by elimination, there results

$$\frac{dy_0}{dx_0} . \delta x_0 - \delta y_0 = 0, \text{ and } \frac{dy_1}{dx_1} . \delta x_1 - \delta y_1 = 0,$$

from these equations, and from the equation of limits $\delta U_1 - \delta U_0 = 0$, which must contain δx_1, δx_0, δy_1, δy_0, the values of δx_0, δy_0 must be eliminated, and the resulting equation will determine the conditions of the problem that must be fulfilled at the limits.

The same conclusion will result, if, instead of an elimination, we multiply the equations $\delta L = 0$, $\delta M = 0$ by two indeterminate quantities λ, μ, and then add them to the equation of limits, which will become

$$\delta U_1 - \delta U_0 + \lambda \delta L + \mu \delta M = 0.$$

This equation will contain terms affected with δx_0, δx_1, δy_0, δy_1, the coefficients of which are separately to be made equal nothing.

The origin of the abscissas and ordinates (when the discussion is concerning curves) is supposed to remain constant; but, it may be supposed to vary, which will be equivalent to the hypothesis of making quantities, such as a, b, contained in V, to vary. In that case, if

$$\delta V = M\delta x + N\delta y + P\delta p + \&c. + A\delta a + B\delta b + \&c.$$

we have, as $M = \frac{dV}{dx}$, and $N = \frac{dV}{dy}$ &c. $A = \frac{dV}{da}$, and $B = \frac{dV}{db}$; and consequently the variation $\delta\int V\,dx$ will $=$

$$\int\delta y.dx\left(N - \frac{dP}{dx} + \frac{d\,Q}{dx^2} - \&c.\right)$$
$$+\int\delta a dx.A + \int\delta b dx.B$$
$$+\delta y\left(P - \frac{dQ}{dx} + \&c.\right)$$
$$+ \&c.$$

now the equation to the curve depends on the equation $N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} -$ &c. and the additional terms $\delta a\int A\,dx$, $\delta b\int B dx$, or $\delta a\int\frac{dV}{da}dx$, $\delta b\int\frac{dV}{db}dx$, will affect the equation of limits.

If $x = x' - a$, then $\delta x = \delta x' - \delta a$; therefore, if x' be the same, $\delta x = -\delta a$; consequently the term $A\delta a$, introduced by reason of the variation of the origin of the co-ordinates, will be affected with a negative sign.

The methods described in this Chapter comprehend all cases belonging to absolute maxima and minima: they extend also, if we employ Euler's reasoning *, to relative maxima and minima; or independently of the

---

* See p. 78.

last author's method, which may be thought not perfectly satisfactory, they may be extended by the following process, which is Lagrange's.

Let u be the function, the integral of which is, to have, within the assigned limits, a determinate value. Let s be its integral, then $ds - u\,dx = 0$; consider this to be an equation of condition, such as $L = 0$; then, since $\lambda\delta L$ was added [p. 107.] to the variation of $\int V\,dx$, we must now add

$$\lambda\delta(ds - udx) \ldots\ldots [a]$$

the first of this, $\lambda\delta ds = \lambda d\delta s = d(\lambda\delta s) - d\lambda.\delta s$; a term therefore $-d\lambda.\delta s$ will be introduced under the integral sign, which, since δs is an arbitrary quantity, must $= 0$; $\therefore\ d\lambda = 0$, or $\lambda = a$, a constant quantity. The term $d(\lambda\delta s)$ will solely affect the equation of limits, and since the whole variation must be taken between s_1 and s_0, $a\delta s_1 - a\delta s_0$ must be added to the equation of limits, which, by the hypothesis of s having a determinate value, must $= 0$.

Hence there remains of [a] l. 10, only $-a\delta(udx)$; consequently, the whole variation is reduced to this $\delta\int(V - au)\,dx$: whence the rule before given, p. 78, which reduces relative maxima and minima to absolute, is derived; for the formula directs us to find the conditions of the absolute maximum or minimum of $\int(V - au)\,dx$, a being a constant quantity.

We now proceed to the last Chapter of this Work, in which, the formulæ previously established, will, with some slight alteration, be applied to the solution of problems.

CHAP. VIII.

Particular Formulæ deduced from the General one, for the Purpose of facilitating the Solution of Problems—Problems solved.

EULER in his treatise has deduced from his general formula, several subordinate ones, limited indeed, but materially expediting the solution of problems. These will be first described.

In the general formula,

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + \&c. \ldots [A]$$

suppose M, Q, and all coefficients excepting N, P, to equal 0; then, $dV = Ndy + Pdp$; but generally [p. 89.] $N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \&c. = 0$ [a]; in this case, therefore, $N - \frac{dP}{dx} = 0$; consequently, $Ndy - dP.\frac{dy}{dx} = 0$; or, since $p = \frac{dy}{dx}$, $Ndy - p.dP = 0$; and substituting, in l. 10, $dV = p.dP + Pdp = d(Pp)$; whence, by integration, $V = Pp + c \ldots\ldots [b]$, c being the correction.

If M is not $= 0$, then $V = \int Mdx + Pp + c \ldots. [c]$.

Let $M = 0$, $N = 0$, and all the coefficients after Q; then $dV = Pdp + Qdq$, but the general formula [a] l. 11. in this case becomes

$$-\frac{dP}{dx} + \frac{d^2Q}{dx^2} = 0;$$

whence, $P = \int \frac{d^2Q}{dx^2} dx + c = \frac{dQ}{dx} + c$;

multiply this by dp, and since $\frac{dp}{dx} = q$, we have

$$P.dp = q.dQ + c.dp$$

hence, substituting, in the equation, [*A*] p. 110.

$$dV = q.dQ + Qdq + c.dp = d(Qq) + c.dp$$

and integrating,

$$V = Qq + cp + c' \ldots\ldots [d]$$

c, c', being the corrections.

If M does not $= 0$, we must add, to the above form, the term $\int Mdx$.

If $M = 0$, but N is not $= 0$, that is, if the form be

$$dV = Ndy + Pdp + Qdq$$

then, since $N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} = 0$,

we have, multiplying by dy, which $= pdx$,

$$Ndy - p.dP + p.\frac{d^2Q}{dx^2}.dx = 0.$$

But, $p.\frac{d^2Q}{dx^2} dx = d\left(p \frac{dQ}{dx}\right) - dp.\frac{dQ}{dx}$

$$= d\left(p.\frac{dQ}{dx}\right) - q.\frac{dQ}{dx}.dx.$$

Hence, substituting, in l. 12.

$$dV = pdP + Pdp - d\left(p.\frac{dQ}{dx}\right) + qdQ + Qdq;$$

consequently, $V = Pp + Qq - p.\frac{dQ}{dx} + c \ldots [e]$.

If $dV = Pdp + Rdr$,

then the formula [a] p. 110. is reduced to

$$-\frac{dP}{dx}-\frac{d^3R}{dx^3}=0;$$

consequently, $P=-\frac{d^2R}{dx^2}+c$, and $Pdp=c.dp-dp.\frac{d^2R}{dx^2}$; hence, since $dp=q.dx$, we have, by substitution,

$$dV=c.dp+Rdr-q.\frac{d^2R}{dx^2}.dx$$

$$=c.dp+Rdr-d\left(q\frac{dR}{dx}\right)+dq.\frac{dR}{dx}$$

[since $dq=rdx$] $=c.dp+Rdr+rdR-d\left(q.\frac{dR}{dx}\right)$

Hence integrating, and adding the correction c',

$$V=cp+Rr-q.\frac{dR}{dx}+c' \quad .\ .\ .\ .\ [f]$$

These forms are sufficient for the solution of the following problems.

PROB. 1.

Required the relation of x and y, such that $\int(ax-y^2).ydx$ shall be a maximum or minimum.

Comparing this expression with $\int Vdx$, $V=axy-y^3$; consequently, $dV=ay.dx+(ax-3y^2)dy$; which, compared with the general formula [A] p. 110. that is, with $dV=Mdx+Ndy+Pdp+Qdq+$&c. gives $M=ay$, $N=ax-3y^2$, $P=0$, $Q=0$; consequently, since $N-\frac{dP}{dx}+$ &c. $=0$, in this case

$$ax-3y^2=0, \text{ and } y=\sqrt{\frac{ax}{3}}.$$

PROB. 2.

Required the shortest curve that can be drawn between two points, or between two curves.

Here $\int ds$ or $\int dx\sqrt{(1+p^2)}=$ a minimum; $\therefore V=\sqrt{(1+p^2)}$ and $dV=\dfrac{pdp}{\sqrt{(1+p^2)}}$; M, N, &c. $=0$: and $P=\dfrac{p}{\sqrt{(1+p^2)}}$; consequently, the formula [a] p. 110, becomes $-d\left(\dfrac{p}{\sqrt{1+p^2}}\right)=0$; $\therefore p=a\sqrt{(1+p^2)}$ and p or $\dfrac{dy}{dx}=\dfrac{a}{\sqrt{(1-a^2)}}$, whence, by integration, $y\sqrt{(1-a^2)}=ax+c$, an equation to a right line.

PROB. 3.

Required the curve of quickest descent between two given points.

The time $=\int\dfrac{ds}{\sqrt{y}}=\int\dfrac{\sqrt{(dx^2+dy^2)}}{\sqrt{y}}=\int\dfrac{\sqrt{\left(1+\dfrac{dy^2}{dx^2}\right)}}{\sqrt{y}}.dx$ $=\int\dfrac{\sqrt{(1+p^2)}}{\sqrt{y}}.dx$, which compared with $\int Vdx$ gives $V=\dfrac{\sqrt{(1+p^2)}}{\sqrt{y}}$; whence $dV=-\dfrac{\sqrt{(1+p^2)}}{2y^{\frac{3}{2}}}dy+\dfrac{p}{\sqrt{y}\sqrt{(1+p^2)}}dp$, which compared with Form [A] p. 110, gives $M=0$, $N=-\dfrac{\sqrt{(1+p^2)}}{2y^{\frac{3}{2}}}$, $P=\dfrac{p}{\sqrt{[y(1+p^2)]}}$, $Q=0$. Now by the Form [b] p. 110, $V=Pp+c$; $\therefore$ in this case

$$\frac{\sqrt{(1+p^2)}}{\sqrt{y}} = \frac{p^2}{\sqrt{y}\cdot\sqrt{(1+p^2)}} + c,$$

whence $\frac{1}{\sqrt{y}\sqrt{(1+p^2}} = c$, or $\frac{dx}{\sqrt{y}\sqrt{(dx^2+dy^2)}} = c$

and finally, $dx = \frac{c\sqrt{y}}{\sqrt{(1-c^2y)}}.dy$, an equation to a cycloid. See p. 5.

Prob. 4.

Required the relation of x and y when $\int(x^2+y^2)^n.ds =$ a maximum [see Euler's *Methodus Inveniendi*, &c. p. 52.]

Here $V = (x^2+y^2)^n \times \sqrt{(1+p^2)}$, since $ds = dx\sqrt{(1+p^2)}$ and, $dV =$

$$(2nx\,dx + 2ny\,dy)(x^2+y^2)^{n-1}\sqrt{(1+p^2)} + \frac{(x^2+y^2)^n.p}{\sqrt{(1+p^2)}}dp;$$

$\therefore M = 2nx\,(x^2+y^2)^{n-1}\sqrt{(1+p^2)}$, $N = 2ny\,(x^2+y^2)^{n-1}\sqrt{(1+p^2)}$, and $P = \frac{(x^2+y^2)^n.p}{\sqrt{(1+p^2)}}$.

Here it is more convenient to employ the general formula,

$$N - \frac{dP}{dx} + \&c. = 0,$$

which, since

$$dP = \frac{(x^2+y^2)^{n-1}}{\sqrt{(1+p^2)}}.\left(2npx\,dx + 2npy\,dy + \frac{(x^2+y^2)\,dp}{1+p^2}\right)$$

by reduction becomes

$$\frac{2n(y\,dx - x\,dy)}{x^2+y^2} = \frac{dp}{1+p^2}\;.\;.\;.\;.\;.\;[h],$$

if $\theta =$ arc, whose tangent $= \frac{x}{y}$, then $d\theta = \frac{y\,dx - x\,dy}{x^2+y^2}$;

consequently, integrating the equation [*h*], p. 114, $2n\theta = A + c$, *A* being an arc, whose tangent $= p$, or $\frac{dy}{dx}$

$$\therefore\ \frac{x}{y} = \tan.\ \theta = \tan.\ \frac{1}{2n}\,(A + c).$$

PROB. 5.

Required the curve which by a revolution round its axis generates the solid of least resistance. [see Newton, p. 324, 3ᵈ ed. Euler's *Methodus Inveniendi*, &c. p. 51. Simpson's *Fluxions*, p. 487, Ed. 1750. Emerson's *Fluxions*, p. 183, 3ᵈ ed. Lacroix's, *Calcul. Diff.* tom. II. p. 698.]

The resistance $= \int \frac{y.dy^3}{dx^2 + dy^2}$, which, compared with $\int V dx$, gives

$$V = \frac{yp^3}{1+p^2} \quad \left(\text{since } p = \frac{dy}{dx}\right), \text{ whence}$$

$$dV = \frac{p^3}{1+p^2}.dy + \frac{3yp^2 + yp^4}{(1+p^2)^2}.dp\,;$$

consequently, $M = 0$, $N = \frac{p^3}{1+p^2}$, $P = \frac{3yp^2 + yp^4}{(1+p^2)^2}$.

Hence by the Form [*b*] p. 110, using $-c$ for the correctⁿ,

$$\frac{yp^3}{1+p^2} = \frac{3yp^3 + yp^5}{(1+p^2)^2} - c$$

whence, by reduction,

$$c.(1 + p^2)^2 = 2yp^3, \text{ and } y = \frac{c}{2p^3} + \frac{c}{p} + \frac{cp}{2}\,;$$

but, $pdx = dy = -\frac{3c}{2p^4}dp - \frac{c}{p^2}dp + \frac{c}{2}dp;$

$$\therefore dx = -\frac{3c}{2p^5}dp - \frac{c}{p^3}dp + \frac{c}{2}\cdot\frac{dp}{p}$$

and $x = \frac{3c}{8p^4} + \frac{c}{2p^2} + \frac{c}{2}\cdot\text{hyp. log.}\,p + c'$

which equation must be combined with $c(1+p^2)^2 = 2yp^3$.

PROB. 6.

Required the curve in which $\int yxds$ is a maximum, [see Euler, *Methodus Inveniendi*, &c. p. 52.]

Since, $yxds = yx\sqrt{(1+p^2)}.dx$, $V = yx\sqrt{(1+p^2)}$ and $dV = y\sqrt{(1+p^2)}dx + x\sqrt{(1+p^2)}\,dy + \frac{xyp}{\sqrt{(1+p^2)}}.dp$;

$$\therefore M = y\sqrt{(1+p^2)},\ N = x\sqrt{(1+p^2)},\ P = \frac{xyp}{\sqrt{(1+p^2)}}$$

In this case therefore, we must use the Form [c] p. 110, and then $yx\sqrt{(1+p^2)} = \int y\sqrt{(1+p^2)}.dx + \frac{xyp^2}{\sqrt{(1+p^2)}} + c$

$$\text{or, } \frac{yx}{\sqrt{(1+p^2)}} = c + \int y\sqrt{(1+p^2)}.dx;$$

which by reduction leads to a differential equation of the second order.

PROB. 7.

Required the curve, or in other terms, the relation of x and y when $\int\frac{d^2y}{ds} =$ a maximum. (See Euler, *Comm. Acad. Petrop.* tom. VIII. p. 171.)

$$\frac{d^2y}{ds} = \frac{d^2y}{dx\sqrt{(1+p^2)}} = \frac{d^2y}{dx^2}\cdot\frac{dx}{\sqrt{(1+p^2)}};\ \therefore V = \frac{q}{\sqrt{(1+p^2)}}$$

and $dV = -\frac{qp}{(1+p^2)^{\frac{3}{2}}}.dp + \frac{1}{\sqrt{(1+p^2)}}.dq$;

$\therefore$ $M = 0$, $N = 0$, $P = -\frac{qp}{(1+p^2)^{\frac{3}{2}}}$, $Q = \frac{1}{\sqrt{(1+p^2)}}$;

$\therefore$ by the Form [*d*] p. 111,

$$\frac{q}{\sqrt{(1+p^2)}} = \frac{q}{\sqrt{(1+p^2)}} + cp + c',$$

whence, p or $\frac{dy}{dx} = -\frac{c'}{c}$, and $cy = c'' - c'x$; an equation to a right line.

PROB. 8.

Required the curve in which $\int \frac{d^2y}{y.ds}$ = a maximum. [see *Comm. Acad. Petrop.* tom. VIII. p. 171.]

$$\frac{d^2y}{y.ds} = \frac{d^2y}{dx^2.y\sqrt{(1+p^2)}}.dx; \therefore V = \frac{q}{y\sqrt{(1+p^2)}}$$

and $dV = -\frac{q}{y^2\sqrt{(1+p^2)}}.dy - \frac{qp}{y(1+p^2)^{\frac{3}{2}}}.dp + \frac{1}{y\sqrt{(1+p^2)}}dq$;

$\therefore$ $M = 0$, $N = -\frac{q}{y^2\sqrt{(1+p^2)}}$, $P = -\frac{qp}{y.(1+p^2)^{\frac{3}{2}}}$,

$Q = \frac{1}{y\sqrt{(1+p^2)}}$; $\therefore$ by the Form [*e*] p. 111,

$\left[\text{since } dQ = -\frac{dy}{y^2\sqrt{(1+p^2)}} - \frac{pdp}{y(1+p^2)^{\frac{3}{2}}}\right]$

$$\frac{q}{y\sqrt{(1+p^2)}} = -\frac{qp^2}{y(1+p^2)^{\frac{3}{2}}} + \frac{q}{y\sqrt{(1+p^2)}} + \frac{p^2}{y^2\sqrt{(1+p^2)}} + \frac{qp^2}{y(1+p^2)^{\frac{3}{2}}} - c.$$

consequently, $\frac{p^2}{y^2\sqrt{(1+p^2)}} = c$, and

$$p^2 = \frac{c^2y^4}{2} \pm cy^2\sqrt{\left(1 + \frac{c^2y^4}{4}\right)}$$

and since $p = \frac{dy}{dx}$, $dx = \frac{\sqrt{2}.dy}{y\sqrt{c^2y^2 \pm c\sqrt{(4+c^2y^4)}}}$.

PROB. 9.

Required the curve which, within its own arc, its evolute, and radius of curvature, shall contain the least area*. [see Euler, *Comm. Acad. Petrop.* tom. VIII. p. 169; also *Methodus Inveniendi*, &c. p. 64.]

The radius of curvature $= \frac{ds^3}{-dx.d^2y}$ † $= \frac{(dx^2+dy^2)^{\frac{3}{2}}}{-dx.d^2y}$

$= -\frac{(1+p^2)^{\frac{3}{2}}}{dx.d^2y}dx^3 = -\frac{(1+p^2)^{\frac{3}{2}}}{q}$, since $q = \frac{dp}{dx} = \frac{d^2y}{dx^2}$.

Hence, the differential of the area $= \frac{\text{radius} \times ds}{2} =$

$-\frac{(1+p^2)^{\frac{3}{2}}}{q}\sqrt{(1+p^2)}dx = -\frac{(1+p^2)^2}{q}.dx$, which compared with Vdx, gives $V = -\frac{(1+p^2)^2}{q}$; consequently,

$$dV = -\frac{4p}{q}(1+p^2).dp + \frac{(1+p^2)^2}{q^2}dq;$$

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[*] The area is *AOR* in Simpson's *Diagram*, p. 78; *Fluxions*, ed. 1750.

[†] Simpson, p. 72. Vince, p. 149, First Edit. Woodhouse *Anal. Calc.* p. 180.

$\therefore\ M = 0,\ N = 0,\ P = -\frac{4p}{q}(1+p^2),\ Q = \frac{(1+p^2)^2}{q^2}$

$\therefore$ by Form $[d]$ p. 111, that is, $V = Qq + cp + c'$, we have

$$-\frac{(1+p^2)^2}{q} = \frac{(1+p^2)^2}{q} + cp + c',$$

$$\text{or } -\frac{2(1+p^2)^2}{q} = cp + c'; \text{ but } q = \frac{dp}{dx},$$

$$\therefore\ -2(1+p^2)^{\frac{3}{2}}\sqrt{(1+p^2)}\,.\,dx = cp\,dp + c'.dp;$$

or, since, $ds = \sqrt{(1+p^2)}.dx$

$$-2\,ds = \frac{cp\,dp}{(1+p^2)^{\frac{3}{2}}} + \frac{c'.dp}{(1+p^2)^{\frac{3}{2}}};$$

$$\therefore\ -2s = -\frac{c}{\sqrt{(1+p^2)}} + \frac{c'.p}{\sqrt{(1+p^2)}} + c''$$

$$= \frac{c'p - c}{\sqrt{(1+p^2)}} + c''.$$

If c, $c'' = 0$, $-2s = \frac{c'p}{\sqrt{(1+p^2)}}$, which shews the curve to be a cycloid: for by p. 5. $\frac{dy}{dx}$ or $p = \sqrt{\left(\frac{a-y}{y}\right)}$

$\therefore\ \frac{p}{\sqrt{(1+p^2)}} = \sqrt{\frac{(a-y)}{a}}$, and the arc of a cycloid measured from its vertex $= 2$ chord of generating circle $= 2\sqrt{a(a-y)} = \frac{2ap}{\sqrt{(1+p^2)}}$.

PROB. 10.

Required the curve in which $\int \frac{d^3y}{dx.dy}$ is a minimum.

[see Euler, *Comm. Acad. Petrop.* tom. VIII. p. 185.]

Here $\frac{d^3y}{dx.dy} = \frac{d^3y}{dx^3} \times \frac{1}{p}.dx$; $\therefore V = \frac{r}{p}$

and $dV = -\frac{r}{p^2}.dp + \frac{1}{p}\,dr$;

$\therefore M = 0,\ N = 0,\ P = -\frac{r}{p^2},\ Q = 0,\ R = \frac{1}{p}$

and by the Form [f] p. 112,

$$\frac{r}{p} = cp + \frac{r}{p} + q.\frac{dp}{dx}.\frac{1}{p^2} \pm c'.$$

Hence, $0 = cp + \frac{q^2}{p^2} - c'$, and $q = p\sqrt{(c' - cp)}$

or, $\frac{dp}{dx} = p\sqrt{(c' - cp)}$ and $\therefore dx = \frac{dp}{p\sqrt{(c' - cp)}}$

$$\text{and } x = \frac{1}{\sqrt{c'}}.\text{h. l.}\left(\frac{\sqrt{c'} - \sqrt{(c' - cp)}}{\sqrt{c'} + \sqrt{(c' - cp)}}\right).$$

PROB. 11.

Required the relation of x and y when $\int\frac{(d^2y)^2}{dx^3}$ is a maximum, [see Lacroix, *Diff. Calc.* p. 704 : also Borda *Mem. Acad. des Sciences*, 1767. p. 560.]

$$\frac{(d^2y)^2}{dx^3} = \left(\frac{d^2y}{dx^2}\right)^2.dx = q^2.dx; \quad \therefore V = q^2,$$

and $dV = 2q.dq$; $\therefore M = 0,\ N = 0,\ P = 0,\ Q = 2q,\ R = 0$, &c.

$\therefore$ by the Form [a] p. 110, $\frac{d^2Q}{dx^2} = 0$, or $2.\frac{d^2q}{dx^2} = 0$;

and integrating $2.\frac{dq}{dx} = c$, $2q$ or $2\frac{dp}{dx} = cx + c'$

$$2p \text{ or } 2\frac{dy}{dx} = \frac{cx^2}{2} + c'x + c'', \text{ and}$$

$$2y = \frac{cx^3}{2.3} + \frac{c'x^2}{1.2} + c''x + c'''.$$

The final equation contains four arbitrary quantities, c, c', &c.: in order to determine them, suppose the values of y to be given at two points of the abscissa ($x = 0$, $x = a$), and also the angles in which the tangents to the curve at those points * are inclined to the axis; for the sake of simplicity, suppose $y = 0$, when $x = 0$; and $y = b$, when $x = a$: also the values of the two tangents of the angles in which the curve cuts the axis to be t, t': then we have, for the determination of c, c', &c. the following equations:

$$x = 0,\ y = 0; \quad \therefore\ c''' = 0,$$

$$x = a,\ y = b; \quad \therefore\ 2b = \frac{ca^3}{2.3} + \frac{c'.a^2}{2} + c''a,$$

$$x = 0;\ \therefore \frac{dy}{dx} = t; \quad \therefore\ 2t = c'',$$

$$x = a;\ \therefore\ \frac{dy}{dx} = t'; \quad \therefore\ 2t' = \frac{ca^2}{2} + c'a + c'',$$

whence the values of c, c'.

The preceding problems involve one property only; that of the maximum or minimum; and therefore, in strictness, ought not to be classed amongst Isoperimetrical problems, since they involve neither the Isoperimetrical property, properly so called, nor any other equally affecting the theory and the analytical processes. The following problems involve more than one property.

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* This is only one of the many hypotheses that may be framed for the determination of the arbitrary quantities.

PROB. 12.

Required the brachystochrone, that is, the curve of quickest descent, when the length of the curve is given. [See John Bernoulli's Works; tom. II. p. 255. *Acad. des Sciences,* 1718. p. 120; also of this Work, pp. 18, &c.]

Here, by the formulæ, pp. 78, 109, we must find the variation of $\int dx\ (V - au)$, putting $V - au$ instead of V: now $V = \frac{\sqrt{(1+p^2)}}{\sqrt{y}}$ and $u = \sqrt{(1+p^2)}$;

$$\therefore V - au, \text{ or } V' = \frac{\sqrt{(1+p^2)}}{\sqrt{y}} - a\sqrt{(1+p^2)};$$

$$\therefore dV' = -\frac{1}{2y^{\frac{3}{2}}}\sqrt{(1+p^2)}.dy + \left(\frac{1}{\sqrt{y}} - a\right)\frac{p}{\sqrt{(1+p^2)}}.dp;$$

$$\therefore N = -\frac{1}{2y^{\frac{3}{2}}}\sqrt{(1+p^2)},\ P = \left(\frac{1}{\sqrt{y}} - a\right)\times\frac{p}{\sqrt{(1+p^2)}}$$

$\therefore$ by the Form [*b*] p. 110.

$$\sqrt{(1+p^2)}\left(\frac{1}{\sqrt{y}} - a\right) = \frac{p^2}{\sqrt{(1+p^2)}}\left(\frac{1}{\sqrt{y}} - a\right) + c;$$

or $\left(\frac{1}{\sqrt{y}} - a\right) = c\sqrt{(1+p^2)}$, and by reduction,

$$dx = \frac{c\sqrt{y}.dy}{\sqrt{[1 - 2a\sqrt{y} + (a^2 - c^2)y]}}.$$

If, instead of the length, the area had been given, then $\int u dx = \int y dx$; and consequently, we should have

$$V', \text{ or } V - au = \frac{\sqrt{(1+p^2)}}{\sqrt{y}} - ay,$$

in which case, $P = \frac{p}{\sqrt{y}\sqrt{(1+p^2)}}$, and, $\frac{\sqrt{(1+p^2)}}{\sqrt{y}} - ay = \frac{p^2}{\sqrt{y}\sqrt{(1+p^2)}} + c,$

whence, $\frac{1}{\sqrt{y}\sqrt{(1+p^2)}} = c + ay,$ and by reduction,

$$dx = \frac{(c+ay)\sqrt{y}.dy}{\sqrt{[1-y(c+ay)^2]}}.$$

If $c=0$, $dx = \frac{ay^{\frac{3}{2}}dy}{\sqrt{(1-a^2y^3)}}$ which agrees with Simpson's result, *Fluxions*, p, 492. But, since c is not necessarily $=0$, it follows, that Simpson's solution is not so general as it ought to be.

PROB. 13.

Required the solid of least resistance amongst all the solids of equal capacity. [See Maclaurin's *Fluxions*, p. 751. Emerson's *Fluxions*, p. 188.]

Here, [see Prob. 5. p. 155.], $V = \frac{yp^3}{1+p^2}$, and $\int u\,dx = \int \pi y^2 dx$;

$\therefore V - au$; or $V' = \frac{yp^3}{1+p^2} - ay^2$ (including π in the quantity a.)

Hence, $P = \frac{3yp^2+yp^4}{(1+p^2)^2}$, and by the Form [$b$] p. 110,

$\frac{yp^3}{1+p^2} - ay^2 = \frac{3yp^3+yp^5}{(1+p^2)^2} - c$, whence, by reduction,

$(c-ay^2)(1+p^2)^2 = 2yp^3$, or $(c-ay^2)\,ds^4 = 2y\,dy^3.dx$.

If instead of the condition of equal capacity, that of equal superficies be substituted, we have

$$V-au, \text{ or } V'=\frac{yp^3}{1+p^2}-ay\sqrt{(1+p^2)},$$

$$\text{and } P=\frac{3yp^2+yp^4}{(1+p^2)^2}-\frac{ayp}{\sqrt{(1+p^2)}},$$

whence, by the Form [*b*] p. 110,

$$\frac{yp^3}{1+p^2}-ay\sqrt{(1+p^2)}=\frac{3yp^3+yp^5}{(1+p^2)^2}-\frac{ayp^2}{\sqrt{(1+p^2)}}-c,$$

and, by reduction,

$$c.(1+p^2)^2=2yp^3+ay.(1+p^2)^{\frac{3}{2}}$$

$$\text{or } c.ds^4=2y.dy^3.dx+ay.ds^3.dx.$$

If the solid of least resistance be required with both the specified conditions, then we must use a formula $\int(V-au-b\nu)\,dx$. See p. 78, and if $u=y^2$, and $\nu=y\sqrt{(1+p^2)}$, we have

$$V-au-b\nu, \text{ or } V'=\frac{yp^3}{1+p^2}-ay^2-by\sqrt{(1+p^2)},$$

and consequently, by the Form [*b*] p. 110.

$$V'=\frac{3yp^3+yp^5}{(1+p^2)^2}-\frac{byp^2}{\sqrt{(1+p^2)}}-c, \text{ whence}$$

$$c(1+p^2)^2=2yp^3+by(1+p^2)^{\frac{3}{2}}+ay^2(1+p^2)^2,$$

$$\text{or } (c-ay^2)ds^4=2ydy^3dx+byds^3.dx.$$

PROB. 14.

Given the length of the curve, to determine its nature when the solid generated by its rotation is a maximum. [See Simpson, p. 486: Maclaurin's *Fluxions*, p. 749; Euler, *Methodus Inveniendi*, &c. p. 196.]

$\int Vdx=\int\pi y^2.dx$ [$\pi=3.14159$ &c.] and $\int udx=\int\sqrt{(1+p^2)}dx$;

$$\therefore V', \text{ or } V-au=\pi y^2-a\sqrt{(1+p^2)},$$

and $dV' = 2\pi y.dy - \frac{ap.dp}{\sqrt{(1+p^2)}}$, whence $P = -\frac{ap}{\sqrt{(1+p^2)}}$;

$\therefore$ by the Form [b] p. 110.

$$\pi y^2 - a\sqrt{(1+p^2)} = c - \frac{ap^2}{\sqrt{(1+p^2)}},$$

and $\therefore \sqrt{(1+p^2)} = \frac{a}{\pi y^2 - c}$, and by reduction,

p, or $\frac{dy}{dx} = \frac{\sqrt{[a^2 - (\pi y^2 - c)^2]}}{\pi y^2 - c}$, or $dx = \frac{(\pi y^2 - c)}{\sqrt{[a^2 - (\pi y^2 - c)^2]}} dy$.

If $c = 0$, $dx = \frac{\pi y^2}{\sqrt{(a^2 - \pi^2 y^4)}} dy$, which is Simpson's result, and, like the preceding, restricted.

Euler says, that the curve is the elastic; and the curve determined by Simpson is the rectangular elastic curve.

Prob. 15.

Required the curve that generates the solid of the least surface, the area being given. [See Euler, *Methodus Inveniendi*, &c. p. 198.]

$$V = 2\pi y\sqrt{(1+p^2)}, \quad u = y;$$

$$\therefore V', \text{ or } V - au = 2\pi y\sqrt{(1+p^2)} - ay,$$

and $dV' = [2\pi\sqrt{(1+p^2)} - a]\,dy + \frac{2\pi yp}{\sqrt{(1+p^2)}}.dp$,

whence $P = \frac{2\pi yp}{\sqrt{(1+p^2)}}$; $\therefore$ by the Form [b] p. 110,

$$2\pi y\sqrt{(1+p^2)} - ay = \frac{2\pi yp^2}{\sqrt{(1+p^2)}} + c,$$

and, by reduction,

$$p, \text{ or } \frac{dy}{dx} = \frac{\sqrt{[4\pi^2y^2 - (c+ay)^2]}}{c+ay},$$

$$\text{or } dx = \frac{c+ay}{\sqrt{[4\pi^2y^2 - (c+ay)^2]}}.dy.$$

If instead of the area, the length is given, then $\int u dx = \int ds = \int dx \sqrt{(1+p^2)}$; and $u = \sqrt{(1+p^2)}$; $\therefore\ V' = 2\pi y \sqrt{(1+p^2)} - a\sqrt{(1+p^2)}$, and consequently

$$P = \frac{2\pi yp - ap}{\sqrt{(1+p^2)}} \text{ and by the Form [}b\text{] p. 110,}$$

$$(2\pi y - a)\sqrt{(1+p^2)} = \frac{(2\pi y - a)p^2}{\sqrt{(1+p^2)}} + c,$$

whence, by reduction,

$$dx = \frac{c\,dy}{\sqrt{[(2\pi y - a)^2 - c^2]}},$$

and $x = \frac{c}{2\pi}$.hyp. log. $[(2\pi y - a) + \sqrt{(2\pi y - a)^2 - c^2}] + c'$, and the curve is the catenary. [See Euler, *Methodus Inveniendi*, p. 198.]

PROB. 16.

Given the length of a curve; required its nature, when its area is a maximum. [See p. 41.]

Here $V' = \sqrt{(1+p^2)} - ay$; $\therefore P = \frac{p}{\sqrt{(1+p^2)}}$, and by the Form [*b*] p. 110,

$$\sqrt{(1+p^2)} - ay = \frac{p^2}{\sqrt{(1+p^2)}} + c;\ \therefore\ \frac{1}{\sqrt{(1+p^2)}} = c + ay,$$

and p, or $\frac{dy}{dx} = \frac{\sqrt{[1-(c+ay)^2]}}{(c+ay)}$, hence,

$dx = \frac{c+ay}{\sqrt{[1-(c+ay)^2]}}.dy$, and $x = c' - \sqrt{[1-(c+ay)^2]}$ [illegible] equation to a circle.

PROB. 17.

If from a point two straight lines be drawn, inflected to each other, and their extremities be joined by a curve; it is required to find its nature when its length is a maximum; the area included within the two lines and the curve being given. [See Euler, *Methodus Inveniendi*, &c. p. 136.]

Let a line drawn from the fixed point to the curve, (usually called a radius vector) = y, and let $d\theta$, (rad. = 1,) be the elementary angle between two contiguous lines, y and $y+dy$, then the arc $=\int\sqrt{(dy^2+y^2.d\theta^2)}$. But the differential of the area $=\frac{y^2.d\theta}{2}=\frac{dx}{2}$, dx being invariable; $\therefore$ $d\theta^2=\frac{dx^2}{y^4}$; and substituting

$\int\sqrt{\left(dy^2+\frac{dx^2}{y^2}\right)}$ = a maximum, or putting $p=\frac{dy}{dx}$,

$\int\sqrt{\left(p^2+\frac{1}{y^2}\right)}.dx$ = a maximum. Hence, $V=\sqrt{\left(p^2+\frac{1}{y^2}\right)}$, and $P=\frac{yp}{\sqrt{(y^2p^2+1)}}$,

and by the Form [*b*] p. 110,

$$\sqrt{\left(p^2+\frac{1}{y^2}\right)}=\frac{yp^2}{\sqrt{(y^2p^2+1)}}+c, \text{ or } 1=cy\sqrt{(y^2p^2+1)};$$

Hence p, or $\frac{dy}{dx}=\frac{\sqrt{(1-c^2y^2)}}{cy^2}$; $d\theta=\frac{dx}{y^2}=\frac{c.dy}{\sqrt{(1-c^2y^2)}}$ the property of a circular arc described on a diameter =

$\frac{1}{c}$; the fixed point, from which lines such as y are drawn, being in one extremity of the diameter.

Prob. 18.

Required the curve that, by a revolution round its axis, generates the greatest solid under a given surface. [See Euler, *Methodus Inveniendi*, p. 194.]

Here V', or $V - au = \pi y^2 - 2a\pi y\sqrt{(1+p^2)}$;

$$\therefore P = -\frac{2a\pi yp}{\sqrt{(1+p^2)}},$$ and by the Form [*b*] p. 110,

$$\pi y^2 - 2a\pi y\sqrt{(1+p^2)} = -\frac{2a\pi yp^2}{\sqrt{(1+p^2)}} + c\pi,$$

putting $c\pi$ for the correction: hence, by reduction,

$$p \text{ or } \frac{dy}{dx} = \frac{\sqrt{[4a^2y^2-(y^2-c)^2]}}{y^2-c};$$

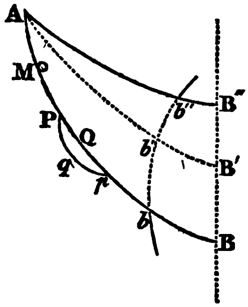
$$\therefore dx = \frac{(y^2-c).dy}{\sqrt{[4a^2y^2-(y^2-c)^2]}}$$ the equation to the curve.

If $c=0$, $dx = \frac{ydy}{\sqrt{(4a^2-y^2)}}$, an equation to a circle: which Simpson [*Fluxions*, p. 487], determines it to be; but, it is plain that this is only a particular case of the general solution. Lyons's *Solution*, p. 100, is also equally restricted with Simpson's.

Prob. 19.

Of all Isochronous curves to find AB such, that the space included between the arc AB, and a chord drawn from A to B shall be a maximum. [See John Bernoulli's Works, tom. II. p. 263; also *Acad. Roy. des Sciences*, 1718, p. 132.]

Here the time, a given quantity represented by $\int u dx$, answers to $\int dx \frac{\sqrt{(1+p^2)}}{\sqrt{y}}$,



and $\int V dx$, the maximum, to

$$\int\left[y dx - d\left(\frac{xy}{2}\right)\right], \text{ or } \int\left(\frac{y dx}{2} - \frac{x dy}{2}\right),$$

or $\int\left(\frac{y}{2} - \frac{xp}{2}\right) dx$, since $p = \frac{dy}{dx}$.

Hence, $V - au$, or $V' = \frac{1}{2}(y - xp) - a\frac{\sqrt{(1+p^2)}}{\sqrt{y}}$;

$\therefore M = -\frac{p}{2}$, $N = \frac{1}{2} + \frac{a}{2}\frac{\sqrt{(1+p^2)}}{y^{\frac{3}{2}}}$, and $P = -\frac{x}{2} - \frac{ap}{\sqrt{y}\sqrt{(1+p^2)}}$;

∴ by the Form [c] p. 110,

$$\frac{1}{2}(y - xp) - a\frac{\sqrt{(1+p^2)}}{\sqrt{y}} = -\frac{xp}{2} - \frac{ap^2}{\sqrt{y}.\sqrt{(1+p^2)}} - \int\frac{p dx}{2} + c.$$

But $p = \frac{dy}{dx}$; $\therefore \int p dx = y$. Hence, by reduction,

$$y - \frac{a}{\sqrt{y}\sqrt{(1+p^2)}} = c, \text{ and } p, \text{ or } \frac{dy}{dx} = \frac{\sqrt{[a^2 - y(y-c)^2]}}{\sqrt{y}.(y-c)},$$

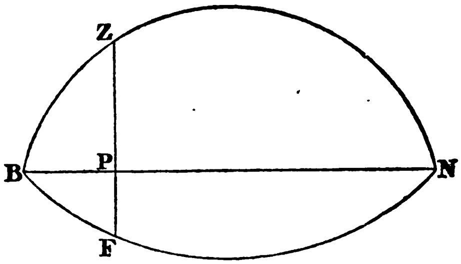
and consequently,

$$dx = \frac{\sqrt{y}.(y-c)dy}{\sqrt{[a^2 - y.(y-c)^2]}},$$

the equation to the curve.

PROB. 20.

Of all Isoperimetrical curves drawn between *B* and *N*, to find *BFN* such, that *BZN* shall contain the greatest area, *PZ* being a function of *PF*. [See James and John Bernoulli's Works, p. 909, and p. 816; also pp. 12, 25, of this Work.]



Let $PF = y$, and let $PZ =$ a function of $y = Y$; then $\int V dx$ corresponds to $\int Y dx$, and $\int u dx$ to $\int \sqrt{(1+p^2)}.dx$.

Hence, $V + au$, or $V' = Y + a\sqrt{(1+p^2)}$ and $dV' = \frac{dY}{dy}dy + \frac{ap}{\sqrt{(1+p^2)}}.dp$; $\therefore P = \frac{ap}{\sqrt{(1+p^2)}}$

Hence by the Form [*b*] p. 110,

$$Y + a\sqrt{(1+p^2)} = \frac{ap^2}{\sqrt{(1+p^2)}} + c,$$

or, $\frac{a}{\sqrt{(1+p^2)}} = c - Y$, and by reduction,

p, or $\frac{dy}{dx} = \frac{\sqrt{[a^2-(c-Y)^2]}}{c-Y}$, and consequently,

$$dx = \frac{(c-Y)dy}{\sqrt{[a^2-(c-Y)^2]}}$$ the equation to the curve.

In the equation, as it stands, there are two arbitrary undetermined quantities a and c. The integration of the equation would introduce a third: and to determine these three, we have given, the two points B and N, and the length of the curve.

PROB. 21.

Required the curve which generates by its rotation the solid of the greatest volume; the length of the curve and its area being given. [See Lacroix, *Calc. Diff. et Int.* vol. II. p. 713.]

Here we must use the form [See p. 78.]

$\int(V - au - bv).dx$, corresponding to which is

$$\int[y^2 - ay - b.\sqrt{(1+p^2)}]\,dx,$$

a and b involving, in their values, π and other given quantities.

Hence, the differential of the quantity corresponding to V is

$$(2y - a).dy - bp.\frac{dp}{\sqrt{(1+p^2)}};$$

∴ by the Form [b] p. 110,

$$y^2 - ay - b\sqrt{(1+p^2)} = c - \frac{bp^2}{\sqrt{(1+p^2)}};$$

$$\therefore y^2 - ay - c = \frac{b}{\sqrt{(1+p^2)}},$$ and by reduction,

$$p, \text{ or } \frac{dy}{dx} = \frac{\sqrt{[b^2-(y^2-ay-c)^2]}}{y^2-ay-c}, \text{ and}$$

$$dx = \frac{(y^2-ay-c)\,dy}{\sqrt{[b^2-(y^2-ay-c)^2]}},$$

an equation to the elastic curve.

PROB. 22.

Let now PZ, instead of being a function of y, (see Fig. p. 130.) be a function of the arc BF (s). [See James and John Bernoulli's Works, p. 912, and p. 917: *Maclaurin's Fluxions*, p. 508, also p. 25. of this Work.]

Here $\int V dx$ corresponds to $\int Z dx$, Z a function of s, and $\int u dx$ corresponds to $\int dx\sqrt{(1+p^2)}$;

$$\therefore V - au, \text{ or } V' = -a\sqrt{(1+p^2)} + Z,$$

$$\text{and } dV' = -\frac{ap}{\sqrt{(1+p^2)}}.dp + \frac{dZ}{ds}.ds.$$

This form therefore is to be solved by the method given in p. 93: and, on comparison, we have $\mu = 0$, $\nu = 0$, $\pi = -\frac{ap}{\sqrt{(1+p^2)}}$, $T = \frac{dZ}{ds}$, $ds = dt$. The formula of solution then, p. 93, is

$$ad\left(\frac{p}{\sqrt{(1+p^2)}}\right) + d\left(\frac{kp}{\sqrt{(1+p^2)}}\right) = 0;$$

k being $= H - \int T dx$. Hence, integrating

$$\frac{ap+kp}{\sqrt{(1+p^2)}} = c, \text{ or } ap + kp = c\sqrt{(1+p^2)}$$

$$\therefore k = \frac{c\sqrt{(1+p^2)} - ap}{p}, \text{ and } dk = \frac{cdp}{p^2\sqrt{(1+p^2)}} = -Tdx;$$

$$\therefore Tdx\sqrt{(1+p^2)}, \text{ or } Tds = -c.\frac{dp}{p^2}: \therefore \int Tds, \text{ that is,}$$

$\int dZ$ or $Z = c' + \frac{c}{p}$. Hence p or $\frac{dy}{dx} = \frac{c}{Z - c'}$, and

$$dx = dy\frac{(Z-c')}{c}.$$

If dx, dy are to be expressed in terms of the arcs, then since $dx = dy.\frac{(Z-c')}{c}$, $ds = \sqrt{\left[1 + \left(\frac{Z-c'}{c}\right)^2\right]}.dy$,

consequently, $dy = \frac{c.ds}{\sqrt{[c^2 + (Z-c')^2]}}$

$$dx = \frac{(Z-c').ds}{\sqrt{[c^2 + (Z-c')^2]}}.$$

PROB. 23.

Given the length of the curve; required its nature when its center of gravity is most remote from the axis (y). [See Euler, *Comm. Acad. Petrop.* tom. VI. p. 146: Simpson's *Fluxions*, p. 498.

Suppose y to be horizontal, and x vertical, then the distance of the center of gravity from y, $= \frac{\int x ds}{\int ds}$ *.

But $\int ds = s = b$ by hypothesis;

$$\therefore\ V - au, \text{ or } V' = \frac{x.\sqrt{(1+p^2)}}{b} - a\sqrt{(1+p^2)},$$

$$\text{and } P = \left(\frac{x}{b} - a\right) \times \frac{p}{\sqrt{(1+p^2)}}.$$

Hence, by the general formula, $N - \frac{dP}{dx} + \&c.$ N, Q,

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* Simpson's *Fluxions*, p. 205. ed. 1750.

&c. being = 0, we have

$$d\left[\left(\frac{x}{b} - a\right)\frac{p}{\sqrt{(1+p^2)}}\right] = 0;$$

$$\therefore \frac{x}{b} - a = c\frac{\sqrt{(1+p^2)}}{p}, \text{ and } dx = bc \times \frac{-dp}{p^2\sqrt{(1+p^2)}}.$$

Hence, $dx\sqrt{(1+p^2)}$, or $ds = -cb.\frac{dp}{p^2}$, and integrating,

$s = \frac{cb}{p} + c'$, or $p = \frac{cb}{s-c'}$; $\therefore$ since $\sqrt{(1+p^2)} = \frac{ds}{dx}$,

$$dx = \frac{(s-c')\,ds}{\sqrt{[(s-c')^2 + c^2b^2]}},$$

$$\text{and } dy = \frac{cb.ds}{\sqrt{[(s-c')^2 + c^2b^2]}},$$

equations to the catenary.

If the condition of equal length be omitted, that is, if $a = 0$, P will equal $\frac{x}{b}\frac{p}{\sqrt{(1+p^2)}}$; and the nature of the resulting curve will still remain the same, s being represented by b: and if s be not so represented, but be supposed to vary, still the resulting curve will be the catenary, as will appear, with the reason thereof, in the following problem.

PROB. 24.

Suppose a chain or cord, of variable thickness, to be attached to two fixed points; it is required to find the nature of the curve which it ought to form, so that its center of gravity shall be the lowest.

Let the weight of an element of the curve be expressed by $\theta.ds$, θ being a function of s dependent on the law of the chain's thickness; then, the distance of the center of gravity, instead of being $\frac{\int x ds}{\int ds}$, is $\frac{\int x\theta.ds}{\int\theta.ds}$, which is to be a maximum.

Hence, $\delta\left(\frac{\int x\theta ds}{\int\theta ds}\right) = \frac{\int\theta.ds.\delta\int x\theta ds - \int x\theta.ds.\delta\int\theta.ds}{(\int\theta.ds)^2} = 0$,

now $\delta\int x\theta.ds = \delta\int x\theta\sqrt{(1+p^2)}.dx$; $\therefore$ $V = x\theta\sqrt{(1+p^2)}$ and $dV =$

$$\theta\sqrt{(1+p^2)}.dx + \frac{\theta xp.dp}{\sqrt{(1+p^2)}} + x.\sqrt{(1+p^2)}.\frac{d\theta}{ds}.ds.$$

Hence by the Form [*C*] p. 93, supposing $\delta x = 0$, and not regarding *S*, the sum of the definite terms,

$$\begin{aligned}\delta\int Vdx &= \int dx.\delta y\left[-d\left(\frac{\theta xp}{\sqrt{(1+p^2)}}\right)\frac{1}{dx}\right] \\ &+ \int dx.\delta y\left[-d\left(\frac{kp}{\sqrt{(1+p^2)}}\right)\frac{1}{dx}\right]\end{aligned} \qquad [m]$$

since $\nu = 0$, $\pi = \frac{\theta xp}{\sqrt{(1+p^2)}}$, $N = 0$, $P = \frac{p}{\sqrt{(1+p^2)}}$, and *k*, which equals $H - \int Tdx$, being in this case $= H - \int x\sqrt{(1+p^2)}.\frac{d\theta}{ds}.dx = H - \int x.d\theta = H - x\theta + \int\theta.dx$ substituting this value in the expression [*m*]; the equation becomes

$$\delta\int Vdx = \int dx.\delta y.\left[-d\left(\frac{Hp + p\int\theta.dx}{\sqrt{(1+p^2)}}\right).\frac{1}{dx}\right].$$

With regard to the other variation $\delta\int\theta.ds$, *V* answers to $\theta.\sqrt{(1+p^2)}$, and $dV = \frac{\theta pdp}{\sqrt{(1+p^2)}} + \sqrt{(1+p^2)}.\frac{d\theta}{ds}ds$;

∴ by Form [*C*] p. 93, supposing $\delta x = 0$, and not regarding S the sum of the definite parts, $\delta\int V dx =$

$$\int dx.\delta y\left[-d.\frac{\theta p}{\sqrt{(1+p^2)}}\,\frac{1}{dx}-d.\frac{kp}{\sqrt{(1+p^2)}}\,\frac{1}{dx}\right]\,..\,[n]$$

ν being $= 0$, $\pi = \dfrac{\theta p}{\sqrt{(1+p^2)}}$, and k in this case being equal to $K - \int\sqrt{(1+p^2)}.\dfrac{d\theta}{ds}.dx$; that is, $K - \int d\theta$, or $K - \theta$; K is analogous to H, and equals the entire value of θ between the two limiting values of x. Hence, substituting in [n] l. 3, we have

$$\delta\int\theta.ds = \int dx.\delta y\left[-d\left(\frac{Kp}{\sqrt{(1+p^2)}}\right)\frac{1}{dx}\right].$$

If we now substitute in the expression for $\delta\left(\dfrac{\int x\theta.ds}{\int\theta ds}\right)$ p. 135. l. 4, and represent the two integrals of $x\theta\, ds$, and θds, taken between the two limiting values [x_0 and x_1] of x, by A and B, there results

$$B\int dx\delta y\left[-d\left(\frac{Hp+p\int\theta\,dx}{\sqrt{(1+p^2)}}\right).\frac{1}{dx}\right]-$$

$$A\int dx\,\delta y\left[-d\left(\frac{Kp}{\sqrt{(1+p^2)}}\right).\frac{1}{dx}\right]=0,$$

and putting $\dfrac{A}{B} = \lambda$,

$$d\left(\frac{Hp+p\int\theta dx}{\sqrt{(1+p^2)}}\right)=\lambda.d\left(\frac{Kp}{\sqrt{(1+p^2)}}\right),\text{ and integrating}$$

$$Hp+p\int\theta dx=\lambda Kp+c\sqrt{(1+p^2)},\text{ whence}$$

$H - \lambda K + \int \theta dx = c\frac{\sqrt{(1+p^2)}}{p}$, and taking the differential

$\theta dx = -c\frac{dp}{p^2\sqrt{(1+p^2)}}$, or $\theta dx\sqrt{(1+p^2)} = -c\frac{dp}{p^2}$,

or $\theta ds = -c\frac{dp}{p^2}$; whence integrating

$\int\theta ds = \frac{c}{p} + c'$, and p or $\frac{dy}{dx} = \frac{c}{\int\theta ds - c'}$,

and since $\sqrt{(1+p^2)} = \sqrt{\left(1+\frac{dy^2}{dx^2}\right)} = \frac{ds}{dx}$

we have $dx = \frac{\int\theta ds - c'}{\sqrt{[(\int\theta ds - c')^2 + c^2]}}.ds$,

and $dy = \frac{cds}{\sqrt{[(\int\theta ds - c')^2 + c^2]}}$.

If the chain be uniformly thick, $\theta = 1$ and

$$dx = \frac{(s-c')\,ds}{\sqrt{[(s-c')^2+c^2]}},$$

$$dy = \frac{cds}{\sqrt{[(s-c')^2+c^2]}},$$

which equations are the same, in fact, as those in the preceding problem, [p. 134. l. 67], and belong to the curve called the catenary.

In problem the twenty-third, where s is a given quantity, the solution was effected by means of the common formula $N - \frac{dP}{dx}$ + &c.; and the same formula is sufficient, if instead of the process there used, we institute one similar to that in Prob. 24, p. 134, that is, if we investigate the variation of $\frac{\int x ds}{\int ds} - a\int ds$, which equals

$$\frac{s \times \delta\int x ds - \int x ds \delta\int ds}{s^2} - a\delta\int ds$$

or $\frac{1}{s^2}[s.\delta\int[x\sqrt{(1+p^2)}]dx - (\int x ds + as^2)\delta\int\sqrt{(1+p^2)}dx]$

for, it is manifest, that this expression can be expanded by the common formula, p. 88 *.

The condition however of a variable thickness introduces the quantity θ, a function of s; for instance, instead of $\delta\int x\sqrt{(1+p^2)}.dx$ – &c., we have, [see p. 135. l. 5.] $\delta\int x\theta\sqrt{(1+p^2)}$ – &c.; and therefore when the differential of the quantity corresponding to V is taken, a term $x\sqrt{(1+p^2)}\,\frac{d\theta}{ds}.ds$ is introduced, which obliges us to have recourse to the formula [Cc] p. 93.

The next instance will illustrate the use of Lagrange's formula in his third problem, p. 185, in which, the quantity that is to be a maximum is expressed under the form of a differential equation.

PROB. 25.

Required the curve, down which a body falling, in a resisting medium, shall acquire the greatest velocity.

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[*] Euler, *Comm. Acad.* tom. VI. shews, that the catenary equally results, whether the length of the curve be taken into the computation or not. "Potuisset quidem eadem æquatio multo facilius inveniri, si in $\frac{\int x ds}{s}$ neglexissem denominatorem, quippe per priorem conditionem debet in omnibus curvis esse idem: Verum quia hoc fortuito accidit, &c.

Let z represent the square of the velocity, g the force of gravity, ϱ a function of z, the resistance; and let x be supposed vertical: then,

$$dz - 2g.dx + 2\varrho\sqrt{(1+p^2)}.dx = 0, \ldots [1]$$

[by the principles of mechanics.]

Compare this with the formula p. 94. l. 12, and we have $V = z$, $X = 2g$, $U = 2\varrho\sqrt{(1+p^2)}$ a function of ϱ and p;

$$\therefore dU = \frac{dU}{dp}dp + \frac{dU}{d\varrho}.d\varrho = \frac{2\varrho p}{\sqrt{(1+p^2)}}.dp + 2\frac{d\varrho}{dz}.dz\sqrt{(1+p^2)}$$

$$\therefore d\phi = \frac{2\varrho p}{\sqrt{(1+p^2)}}.dp \text{ and } T = 2\frac{d\varrho}{dz}\sqrt{(1+p^2)}:$$

hence, the equations are,

$$\frac{d\lambda}{dx} - 2\lambda.\frac{d\varrho}{dz}\sqrt{(1+p^2)} = 0 \ldots\ldots [2]$$

$$\text{and } \delta V \text{ or } \delta z = 0 = e^{-\int T dx}\int e^{T\,dx}.\delta\phi = 0,$$

$$\text{or } 2\lambda\varrho\frac{p}{\sqrt{(1+p^2)}} = \frac{1}{\sqrt{a}} \ldots\ldots [3]$$

In equation [2] substitute from [1] the value of dz;

$$\therefore 2gd\lambda = 2\lambda\sqrt{(1+p^2)}.d\varrho + 2\varrho\sqrt{(1+p^2)}.d\lambda$$

$$= 2\sqrt{(1+p^2)}[\lambda d\varrho + \varrho.d\lambda] = 2\sqrt{(1+p^2)}d(\lambda\varrho) =$$

$$2.d[\lambda\varrho\sqrt{(1+p^2)}] - 2\lambda\varrho.\frac{pdp}{\sqrt{(1+p^2)}} = (\text{by } [3])$$

$$2.d[\lambda\varrho\sqrt{(1+p^2)}] - \frac{dp}{\sqrt{a}};$$

Hence, integrating, $2g\lambda = 2\lambda\varrho\sqrt{(1+p^2)} - \frac{p}{\sqrt{a}} + c =$

$$\frac{1+p^2}{p\sqrt{a}} - \frac{p}{\sqrt{a}} + c = \frac{1}{\sqrt{a}.p} + c;$$

and $\therefore$ from [3], $\frac{c}{g}\varrho\frac{p}{\sqrt{(1+p^2)}} + \frac{\varrho}{g\sqrt{a}\sqrt{(1+p^2)}} = \frac{1}{\sqrt{a}}:$

whence, by the solution of a quadratic the value of p.

This problem may be also solved, and readily, by means of the general method and formula given in p. 104, thus: since z is to be a maximum, $\int V dx$ corresponds to

$$\int \frac{dz}{dx} dx;$$

$\therefore dV = d\left(\frac{dz}{dx}\right)$; $\therefore$ by comparison, $\pi = 1$, all the other coefficients being $= 0$; $\therefore Y = 0$, $Z = 0$, $Y' = 0$, $Z' = 1$. Again,

$\int W dx$ corresponds to $\int \left[\frac{dz}{dx} - 2g + 2\varrho . \sqrt{(1+p^2)}\right] dx$

$\therefore$ by comparison,

$$P' = \frac{2\varrho p}{\sqrt{(1+p^2)}}, \quad \nu' = 2\sqrt{(1+p^2)}\frac{d\varrho}{dz}, \quad \pi' = 1;$$

$$\therefore (Y) = \lambda N' - \frac{d(\lambda P')}{dx} = -d\left(\frac{2\lambda\varrho p}{\sqrt{(1+p^2)}}\right)\frac{1}{dx},$$

$$(Y') = \lambda P' = \frac{2\lambda\varrho p}{\sqrt{(1+p^2)}},$$

$$(Z) = \lambda\nu' - \frac{d(\lambda\pi')}{dx} = 2\lambda\sqrt{(1+p^2)}.\frac{d\varrho}{dz} - \frac{d\lambda}{dx},$$

$$(Z') = \lambda\pi' = \lambda.$$

Hence, taking from the expanded form, for $\delta\int V dx + \delta\int \lambda W dx$, or

$\delta\int \frac{dz}{dx} dx + \delta\int \left[\lambda\frac{dz}{dx} - 2g\lambda + 2\lambda\varrho\sqrt{(1+p^2)}\right] dx$, that part which is under the sign $\int$, and on which the nature of the curve depends, we have

$$(Y).\delta\omega + (Z)\,\delta\omega' = 0, \text{ and } (Y)=0,\ (Z)=0,$$

since $\delta\omega$, $\delta\omega'$ are independent of each other; hence,

$$d\left(\frac{2\lambda\varrho p}{\sqrt{(1+p^2)}}\right)=0, \text{ and } 2\lambda\frac{d\varrho}{dz}\sqrt{(1+p^2)}-\frac{d\lambda}{dx}=0$$

as before, p. 139. l. 12, 14.

The following problem will illustrate the use of the general method given in p. 104, in which the maximum or minimum property involves a quantity not expressed except by means of a differential equation; which equation, in the subjoined case, is called an equation of condition.

PROB. 26.

Required the brachystochrone, or the curve of quickest descent, in a medium resisting as any function of the velocity.

Let z represent the square of the velocity, g the force of gravity, ϱ a function of z, the resistance; and let y be horizontal and x vertical: then by the principles of mechanics,

$$dz - 2g.dx + 2\varrho.ds = 0,$$

$$\text{or } \left[\frac{dz}{dx}-2g+2\varrho\sqrt{(1+p^2)}\right]dx = 0$$

the differential equation of condition.

If we compare this with Wdx, p. 104, we have

$$dW = 2.\frac{d\varrho}{dz}\sqrt{(1+p^2)}.dz + d\left(\frac{dz}{dx}\right) + 2\varrho.\frac{pdp}{\sqrt{(1+p^2)}};$$

$$\therefore \nu' = 2.\frac{d\varrho}{dz}\sqrt{(1+p^2)},\ \pi = 1,\ \text{since } d\left(\frac{dz}{dx}\right) = dp',$$

$$\text{and } P' = \frac{2\varrho p}{\sqrt{(1+p^2)}};$$

$$\therefore (Y) = -d\frac{(\lambda P')}{dx} = -d\left(\frac{2\lambda\varrho p}{\sqrt{(1+p^2)}}\right)\frac{1}{dx},$$

$$\text{and } (Z) = \lambda\nu' - d\frac{(\lambda\pi')}{dx} = 2\lambda\sqrt{(1+p^2)}\cdot\frac{d\varrho}{dz} - \frac{d\lambda}{dx}.$$

The equation of minimum corresponding to $\int V dx$ is

$$\int\frac{\sqrt{(1+p^2)}}{\sqrt{z}}.dx;$$

$$\therefore dV = \frac{pdp}{\sqrt{z}\sqrt{(1+p^2)}} - \frac{1}{2}\frac{\sqrt{(1+p^2)}}{z^{\frac{3}{2}}}.dz;$$

and by comparison,

$$N=0,\ P = \frac{p}{\sqrt{z}\sqrt{(1+p^2)}},\ Q=0, \&\text{c. } \nu = -\frac{\sqrt{(1+p^2)}}{2z^{\frac{3}{2}}},$$

$$\text{and } Y = N - \frac{dP}{dx} + \&\text{c.} = -d\left(\frac{p}{\sqrt{z}\sqrt{(1+p^2)}}\right)\frac{1}{dx}$$

$$Z = \nu - \frac{d\pi}{dx} + \&\text{c.} = -\frac{\sqrt{(1+p^2)}}{2z^{\frac{3}{2}}}.$$

Hence, substituting in the expression for $\delta\int V dx + \delta\int(\lambda W)dx$ p. 104, and retaining only the parts under the integral sign, which must separately equal 0, we have

$$Y + (Y) = 0, \text{or} - d\left(\frac{p}{\sqrt{z}\sqrt{(1+p^2)}}\right) - d\left(\frac{2\lambda\varrho p}{\sqrt{(1+p^2)}}\right) = 0,$$

$$Z + (Z) = 0,\ \text{or} - \frac{\sqrt{(1+p^2)}}{2z^{\frac{3}{2}}} + 2\lambda\sqrt{(1+p^2)}.\frac{d\varrho}{dz} - \frac{d\lambda}{dx} = 0.$$

If we put $t = \frac{1}{\sqrt{z}} + 2\lambda\varrho$, and integrate the first of these two equations, it becomes

$$\frac{tp}{\sqrt{(1+p^2)}} = \frac{1}{\sqrt{a}} \;\dots\dots\dots\dots\; [1]$$

the second is $\frac{d\lambda}{dx} - \left(2\lambda\frac{d\varrho}{dz} - \frac{1}{2z^{\frac{3}{2}}}\right)\sqrt{(1+p^2)} = 0 \;..\; [2]$

Add to these $dz - 2g.dx + 2\varrho.dx\sqrt{(1+p^2)} = 0 \;..\; [3]$ and the equation to the curve is to be determined from these three.

From the value of t, $\frac{dt}{dz} = -\frac{1}{2z^{\frac{3}{2}}} + 2\frac{d\lambda}{dz}.\varrho + 2\lambda\frac{d\varrho}{dz}$;

and from this equation substitute the value of $-\frac{1}{2z^{\frac{3}{2}}} + 2\lambda\frac{d\varrho}{dz}$, into [2], and there results

$$\frac{d\lambda}{dx} + \left(2\varrho\frac{d\lambda}{dz} - \frac{dt}{dz}\right)\sqrt{(1+p^2)} = 0.$$

In this, substitute the value of dz derived from [3], and there results (two quantities destroying each other)

$$2g.\frac{d\lambda}{dx} - \frac{dt}{dx}\sqrt{(1+p^2)} = 0 \;\dots.\; [a]$$

but $\frac{dt}{dx}\sqrt{(1+p^2)}.dx = d[t\sqrt{(1+p^2)}] - t\frac{p\,dp}{\sqrt{(1+p^2)}}$

$$= d[t\sqrt{(1+p^2)}] - \frac{dp}{\sqrt{a}},$$

(from the value of t in equation [1];)

hence, the equation [a] is

$2g.\frac{d\lambda}{dx} - d\frac{[t\sqrt{(1+p^2)}]}{dx} + \frac{dp}{dx\sqrt{a}}$; and integrating

$$2g\lambda - t\sqrt{(1+p^2)} + \frac{p}{\sqrt{a}} = c;$$

or, substituting the value of *t* from [1] p. 143,

$$2g\lambda - \frac{(1+p^2)}{p\sqrt{a}} + \frac{p}{\sqrt{a}} = c;$$

$$\text{or, } 2g\lambda = \frac{1}{p\sqrt{a}} + c.$$

Hence, $t = \frac{1}{\sqrt{z}} + 2\lambda\varrho = \frac{1}{\sqrt{z}} + \frac{\varrho}{g}\left(\frac{1}{p\sqrt{a}} + c\right) = \frac{\sqrt{(1+p^2)}}{p\sqrt{a}}$, from equation [1], and consequently,

$$\frac{1}{\sqrt{z}} = \frac{\sqrt{(1+p^2)}}{p\sqrt{a}} - \frac{\varrho}{g}\left(\frac{1}{p\sqrt{a}} + c\right);$$

from this equation obtain the value of dz, which will involve the differentials dp, and $\frac{d\varrho}{dz}.dz$. By means of equation [3], eliminate dz, and there will result a differential equation involving dx and dp, or $\frac{d^2y}{dx^2}.dx$: which will be the equation of solution.

In the case of a void, or non-resisting medium $\varrho = 0$;

$\therefore t = \frac{1}{\sqrt{z}} = \frac{\sqrt{(1+p^2)}}{p\sqrt{a}}$, or $\frac{p}{\sqrt{(1+p^2)}\sqrt{z}} = \frac{1}{\sqrt{a}}$,

or $\frac{dy}{ds\sqrt{z}} = \frac{1}{\sqrt{a}}$, an equation to the cycloid. [See pp. 5, 21.]

The preceding instances are sufficient, it is hoped, fully to illustrate the nature and use of those formulæ of solution invented by Euler, on which the nature of the curve, or the relation of x to y, depends. But hitherto, no illustration has been given of those other formulæ, that, on assigning the initial and final variations, have definite and absolute values *. These formulæ, it has been already said, were invented by Lagrange, and their use will be shewn in the following cases, which may be viewed, as unresolved parts, or undetermined conditions belonging to certain of the preceding problems.

PROB. 27.

If the brachystochrone is to lie between a point and a curve, required the angle in which the cycloid must cut the curve.

By Form [*Aa*] p. 89, and since the quantity under the integral sign $\int,=0$,

$$V_1.\delta x_1 + P_1.\delta\omega_1 = 0;$$

for, the initial point is given ; $\therefore\ \delta x_0,\ \delta y_0 = 0$, and $\delta\omega_0 = \delta y_0 - p_0\delta x_0, = 0$, and since

$$V = \frac{\sqrt{(1+p^2)}}{\sqrt{y}},\ Q=0,\ \text{and}\ P = \frac{p}{\sqrt{y}.\sqrt{(1+p^2)}},$$

[see p.113].

Hence, $(V_1 - P_1 p_1).\delta x_1 + P_1.\delta y_1 = 0$,
but $V = Pp + c$ [p. 110]; $\therefore\ c.\delta x_1 + P_1.\delta y_1 = 0$ . . [*m*].
Let now M (a function of x_1 and y_1)$=0$; then, [see p. 107.]

$$\frac{dM}{dx}.\delta x_1 + \frac{dM}{dy}.\delta y_1 = 0,\ \text{and}\ \frac{dy_1}{dx_1}.\delta x_1 - \delta y_1 = 0.$$

Eliminate δx_1, δy_1 by means of this and of the preceding equation [*m*], and there results

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* P. 89, 90, 91.

$$\frac{dy_1}{dx_1} = -\frac{c}{P_1} = -\frac{V-Pp}{P_1} = -\frac{1}{p_1}.$$

Now $p = \frac{dy}{dx}$, y and x belonging to the cycloid, and p_1 is what $\frac{dy}{dx}$ becomes at the limiting curve, when for x and y, x_1 and y_1 are substituted: but, $\frac{dy}{dx}$ is the tangent of the angle which a tangent line to the curve makes with the axis *, and $\frac{dx}{dy}$ or $\frac{1}{p}$ is the co-tangent of the same angle †, and $\frac{1}{p_1}$ is the co-tangent at the limiting curve: for the same reason $\frac{dy_1}{dx_1}$, is the tangent of the angle which a tangent line to the limiting curve makes with the axis: and since $\frac{dy_1}{dx_1} = -\frac{1}{p_1}$, or since, the tangent of the latter angle equals the co-tangent of the former, one angle is the complement of the other: their sum therefore is equal to a right angle: consequently, the remaining angle of the triangle, or that which the two tangents form, is a right angle; or, the two curves intersect each other at right angles. [See p. 11.]

The same conclusion will follow from the expression for the tangent of the difference of two angles [A, B] the

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* *Princ. Anal. Calc.* p. 173.

† *Trig.* p. 9.

denominator of which expression is $1 + \tan. A. \tan. B$ *, and in the symbols previously employed, $1 + p_1 . \frac{dy_1}{dx_1} = 0$;
∴ the fraction or the tangent $(A - B)$, is ∞; or the difference of the two angles is a right angle.

PROB. 28.

Suppose now the brachystochrone to lie, not between a point and a curve, but between two curves. Required the angles in which it must cut them.

By Form [*Aa*] p. 89.

$$V_1.\delta x_1 - V_0.\delta x_0 + P_1.\delta\omega_1 - P_0.\delta\omega_0 = 0;$$

∴ as before $c\delta x_1 + P_1.\delta y_1 - (c.\delta x_0 + P_0.\delta y_0) = 0$, and this equation, if there be no relation between the two limiting curves, or between x_0, y_0, and x_1, y_1, resolves itself into these two,

$c.\delta x_1 + P_1.\delta y_1 = 0$ [1], and $c.\delta x_0 + P_0.\delta y_0 = 0$ .. [2]

If therefore L, (a function of x_0, y_0) $= 0$, be the equation to the initial limiting curve, by combining it with [2] we shall arrive at the same results as were obtained in p. 145, by combining $M = 0$ with [1]; since the respective equations are similar. Hence, the brachystochrone, which here is a cycloid, must cut the first curve also at right angles. [See *Misc. Taurin.* tom. IV. p. 187.]

In this last case the origin of the abscissas is supposed to be fixed, and consequently, the velocity with which the body leaves the first curve is variable, varying as $\sqrt{y_0}$. If the origin of the abscissas is supposed to vary, then, [see p. 108.] terms such as $-\delta a\int\frac{dV}{da}, -\delta b\int\frac{dV}{db}$, will be introduced into the equation of limits: thus, instead of the differential of the time being $\frac{ds}{\sqrt{y}}$, let it be $\frac{ds}{\sqrt{(y-h)}}$, and let h be a function of a $[x_0]$, and b $[y_0]$ solely; then;

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* Woodhouse's *Trigonometry*, p. 23.

since, $V=\frac{\sqrt{(1+p^2)}}{\sqrt{(y-h)}}$, $\frac{dV}{da}=\frac{\sqrt{(1+p^2)}}{2.(y-h)^{\frac{3}{2}}}.\frac{dh}{da}$, and

$$\frac{dV}{db}=\frac{\sqrt{(1+p^2)}}{2(y-h)^{\frac{3}{2}}}.\frac{dh}{db}.$$

In this case therefore, the equation of limits, at the last point where $x=x_1$, or $=a'$, since $\frac{dh}{da'}=0$, and $\frac{dh}{db'}=0$ [h not involving a' and b'] becomes,

$$c\delta x_1+P_1.\delta y_1=0\,.\,.\,.\,.\,.\,[3]$$

but, the equation of limits, at the first point, becomes

$$c.\delta x_0+P_0.\delta y_0+\left(\frac{dh}{da}\delta x_0+\frac{dh}{db}.\delta y_0\right)\int\frac{ds}{2.(y-h)^{\frac{3}{2}}}=0,\ [4]$$

Since by the gen^l. form^a. $N-\frac{dP}{dx}=0$; $\therefore\ \frac{ds}{2(y-h)^{\frac{3}{2}}}=-dP$, and $\int\frac{ds}{2.(y-h)^{\frac{3}{2}}}=-P+\text{corr.}$ Now at the beginning, the integral $=0$, and $P=P_0$; $\therefore$ corr. $=P_0$. Hence, $\int\frac{ds}{2(y-h)^{\frac{3}{2}}}=P_0-P=P_0-P_1$ [from $x=x_0$ to $x=x_1$] the equation [4] therefore becomes

$$\left[c - \frac{dh}{da}(P_0 - P_1)\right]\delta x_0 + \left[P_0 - \frac{dh}{db}\left(P_0 - P\right)\right]\delta y_0$$
$$= 0.$$

This last result, expressed in the equations [3] and [4], comprehends, as particular cases, the preceding ones p. 146, 147; for if $h = 0$, the equations [3] and [4] become,

$$c.\delta x_1 + P_1.\delta y_1 = 0, \text{ and } c.\delta x_0 + P_0\delta y_0 = 0$$

as before; which equations shew that the cycloid must cut the two curves at right angles. This was Lagrange's original determination of the conditions. [See *Misc. Taurin.* tom. II. p. 180.]

If $h = b$, or $b - e$, [e a constant quantity;) that is, if $y - h$, at the point of departure, $= 0$ or $= e$; or if, in other words, the velocity of the body when it quits the first curve is either nothing or constant; or, if the origin of the abscissas, according to M. Borda's expression, be supposed to be situated in the first curve: then, $\frac{dh}{db} = 1$, and $\frac{dh}{da} = 0$; $\therefore$ the equation [4] becomes

$$c.\delta x_0 + P_1.\delta y_0 = 0. \quad \text{But the equation [3] is}$$
$$c.\delta x_1 + P_1.\delta y_1 = 0;$$

consequently, if from the equations $L = 0$, $M = 0$, we eliminate, as before p. 145, δx_0, δx_1, &c. there will result from the first $\frac{dy_0}{dx_0} = -\frac{c}{P_1}$, and from the second $\frac{dy_1}{dx_1} = -\frac{c}{P_1}$; consequently, $\frac{dy_0}{dx_0} = \frac{dy_1}{dx_1}$, or the tangent at the point of intersection of the cycloid and first curve, is parallel to the tangent, at the intersection of the cycloid and second curve. This is the result which M. Borda first arrived at [*Mem. Acad. des Sciences*, 1767, p. 558,] and which was afterwards confirmed by Lagrange, and Legendre. [*Misc. Taurin.* tom. IV. p. 187; and *Mem. Acad. des Sciences*, 1786, p. 30.]

This completes the solution of the brachystochrone, of which there are six cases, [see pp. 113, 122, 141, 145, 147, 149.] and this curve which first called the attention of mathematicians to the connected doctrine and calculus, has been also the object of their latest researches.

Newton, in the *Acta Eruditorum*, Mai 1697, p. 223 * gave, without proof or the authority of his name, a method of describing the cycloid. But John Bernoulli, from the Work recognised its author: "*ex ungue Leonem*" †.

As additional instances of the use of the equation of limits, we will solve the following problems.

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* See also *Phil. Trans.* No. 224. p. 384: his Works by Horsley, vol. IV. p. 415: and his Opuscula, vol. I. p. 289.

† Quoique l'auteur de cette construction par un exces de modestie ne se nomme pas, nous savons pourtant indubitablement par plusieurs circumstances que c'est le celebre Newton, et quand meme nous ne le saurions point d'ailleurs, ce seroit assez de le connoitre par un echantillon, comme *ex ungue Leonem*, Joann. Bernoulli, *Opera*, tom. I. p. 197.

PROB. 29.

Required the conditions for drawing a right line that is to be, the shortest distance, between two given curves.

1. By Prob. 2. p. 113, the curve was determined to be a straight line; also $V = \sqrt{(1+p^2)}$, and

$$P = \frac{p}{\sqrt{(1+p^2)}}.$$

Hence, by the form, p. 89, the equation of limits becomes

$$V_1.\delta x_1 - V_0.\delta x_0 + P_1.\delta\omega_1 - P_0.\delta\omega_0 = 0,$$

or, since $\delta\omega = \delta y - p.\delta x$ [p. 88.], and

$$V - Pp = \frac{1}{\sqrt{(1+p^2)}}, \text{ we have}$$

$$\frac{1}{\sqrt{(1+p_1^2)}}\delta x_1 + P_1.\delta y_1 - \frac{1}{\sqrt{(1+p_0^2)}}\delta x_0 + P_0.\delta y_0 = 0.$$

If x_1, x_0, &c. are independent of each other, this equation resolves itself into

$$\frac{\delta x_1}{\sqrt{(1+p_1^2)}} + P_1.\delta y_1 = 0, \text{ and } \frac{\delta x_0}{\sqrt{(1+p_0^2)}} + P_0.\delta y_0 = 0.$$

If $M = 0$, and $L = 0$, are the two equations to the two limiting curves, there results, as before, p. 145.

$$\frac{dy_1}{dx_1}.\delta x_1 - \delta y_1 = 0, \quad \frac{dy_0}{dx_0}\delta x_0 - \delta y_0 = 0.$$

Hence, by means of these, and of the two preceding equations, eliminating δx_1, δx_0, &c. there results

$$\frac{dy_1}{dx_1} = -\frac{1}{p_1}, \text{ and } \frac{dy_0}{dx_0} = -\frac{1}{p_0};$$

and consequently, for the reasons assigned in p. 146, the line must intersect each curve at right angles.

PROB. 30.

It is required to investigate the conditions for the curve, down which, a body, in a resisting medium, acquires the greatest velocity. [See p. 138.]

We have, p. 104, the equation of limits,

$$0 = (V + \lambda W).\delta x + [Y' + (Y')] \times (\delta y - p\delta x) + [Z' + (Z')]\,(\delta z - p'\delta x),$$

in which [see p. 140.]

$$V = \frac{dz}{dx},\ W = \frac{dz}{dx} - 2g + 2\varrho\sqrt{(1 + p^2)},$$

$$Y' = 0,\ (Y') = \frac{2\lambda\varrho p}{\sqrt{(1 + p^2)}},\ Z' = 1,\ (Z') = \lambda.$$

Hence, the equation of limits is

$$0 = \left[\frac{dz}{dx}(1 + \lambda) - 2g\lambda + 2\varrho\lambda\sqrt{(1 + p^2)}\right]\delta x +$$

$$\frac{2\lambda\varrho p}{\sqrt{(1 + p^2)}}\,(\delta y - p\delta x) + (1 + \lambda)\,\left(\delta z - \frac{dz}{dx}\right)\delta x,$$

and, by reduction,

$$\left(-2g\lambda + \frac{2\lambda\varrho}{\sqrt{(1 + p^2)}}\right)\delta x + \frac{2\lambda\varrho p}{\sqrt{(1 + p^2)}}.\delta y + (1 + \lambda).\delta z = 0,$$

and if from p. 139, we substitute the values of $2g\lambda$ and $2\lambda\varrho$ there results

$$-c\delta x + \frac{\delta y}{\sqrt{a}} + (1 + \lambda).\delta z = 0,$$

which is the general expression: and consequently, the two equations at the limits, x_0, x_1, become

$$-c\delta x_0 + \frac{1}{\sqrt{a}}.\delta y_0 + (1 + \lambda_0).\delta z_0 = 0,$$

$$-c.\delta x_1 + \frac{1}{\sqrt{a}}.\delta y_1 + (1 + \lambda_1).\delta z_1 = 0.$$

Now z is the square of the velocity; and if the initial velocity be given, $\delta z_0 = 0$; in the second equation, since δz_1 is indeterminate, make $(1 + \lambda_1)\,\delta z_1 = 0$, or $\lambda_1 = -1$, and then the two equations are reduced to

$$-c.\delta x_0 + \frac{1}{\sqrt{a}}.\delta y_0 = 0,$$

$$-c.\delta x_1 + \frac{1}{\sqrt{a}}.\delta y_1 = 0.$$

If $L = 0$, and $M = 0$, are the two equations belonging to the limiting curves; then there results, as before, p. 145.

$$\frac{dy_0}{dx_0}.\delta x_0 - \delta y_0 = 0,$$

$$\frac{dy_1}{dx_1}.\delta x_1 - \delta y_1 = 0.$$

From these two last, and the two preceding, eliminating δx_0, δy_0, &c. there results,

$$\frac{dy_0}{dx_0} = c\sqrt{a}, \quad \frac{dy_1}{dx_1} = c\sqrt{a},$$

consequently, the tangents at the respective points of intersection of the curve, and the two limiting curves, are parallel to each other.

The conditions relative to the curve of quickest descent may be similarly determined. The preceding instances are, however, sufficient to explain the use of Lagrange's determinate formulæ.

The purpose of the present treatise, is now accomplished. There are, however, several curious results deducible from Euler's formula; but, to these, since they are not closely connected with the matter of the preceding pages, the reader is merely referred*.

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* Lagrange, *Fonctions Analy.* p. 217, &c. *Leçons sur le Calcul des Fonctions*, p. 401, &c. Lacroix, *Calcul. Diff.* &c. p. 660, 667, &c. Lexell. *Novi Comm. Petrop.* tom. XV. p. 127, &c.

THE END.

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